Q-Exam. Partial Differential Equations   Fall, 2009

1. (15 points) Solve Burger's equation $u_t + uu_x = 0$ in $(0,\infty)\times\mathbb{R}$ with initial data

$$u(0,x) = u_0(x) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 < x < 1 \\ 0 & x \ge 1. \end{cases}$$

2. (15 points) Consider the initial value problem

$$\begin{cases} u_t = \sin u_x \\ u(0,x) = \frac{\pi}{4}x + \frac{\pi}{6}x^2. \end{cases}$$

Verify that the assumptions of the Cauchy-Kovalevsky theorem are satisfied and obtain the series solution up to third order terms about the origin.

3. (10 points) Let B be the unit open ball in $\mathbb{R}^2$ and

$$u(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B, \end{cases}$$

find the function $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ in distribution sense.

4. (15 points) Let $\Omega \in \mathbb{R}^n$ and $u, v \in C^2(\Omega)\cap C(\overline{\Omega})$, $f \in C^1(\mathbb{R})$ such that $f'(t) \ge 0$, for all $t \in \mathbb{R}$. Assume

$$\begin{cases} \Delta u - f(u) \ge \Delta v - f(v) \text{ on } \Omega \\ u \le v \text{ on } \partial\Omega, \end{cases}$$

show that $u \le v$ on Ω.

5. (15 points) Let B be the unit ball in $\mathbb{R}^3$, and $u \in C^2(B)\cap C(\overline{B})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } B \\ u(x,y,z) = z^2 & \text{if } (x,y,z) \in \partial B,\ z \ge 0, \\ u(x,y,z) = z & \text{if } (x,y,z) \in \partial B,\ z < 0. \end{cases}$$

Find the value of $\int_\Omega u(x)dx$, where $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < (1/4)\}$.

6. (15 points) Show that the initial-boundary value problem

$$\begin{cases} \partial_t^2 u = \partial_x^2 u & (x,t) \in (0,l)\times(0,T),\ T,\ l > 0 \\ u(x,0) = 0, & x \in [0,l] \\ u_x(0,t) - u(0,t) = 0,\ u_x(l,t) + u(l,t) = 0,\ t \in [0,T], \end{cases}$$

has zero solution only.

7. (15 points) Let $Q_L = (0, L) \times (0, T] \subset \mathbb{R}^2$ and $u_L \in C(\overline{Q_L}) \cap C^2(Q_L)$ be a solution of the initial-boundary value problem

$$\begin{cases} \partial_t u = \partial_x^2 u & (x,t) \in Q_L \\ u(0,t) = g(t), \ u(L,t) = 0 & \forall t \in [0,T] \\ u(x,0) = 0, & \forall x \in [0,L], \end{cases}$$

where $g(t) \geq 0$. Show that if $L_1 < L_2$, then $u_{L_1}(x,t) \leq u_{L_2}(x,t)$, for $(x,t) \in Q_{L_1}$.