2009 Ph.D. Qualification Exam

Numerical Analysis

1. **(10%)** Let φ be a real-valued function defined on $[a, b]$. Consider the sequence $x^{(k+1)} = \varphi(x^{(k}$ for $k \geq 0$, being $x^{(0)}$ given. Assume that $\varphi \in C^1([a, b])$, $\varphi([a, b]) \subseteq [a, b]$ and there exists a consta $0 < K < 1$ such that $|\varphi'(x)| \leq K$ for all $x \in [a, b]$. Prove that φ has a unique fixed point α in $[a,$ and the sequence $\{x^{(k)}\}$ converges to α for any choice of $x^{(0)} \in [a, b]$. Moreover, we have

$$\lim_{k \to \infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \varphi'(\alpha).$$

2. **(15%)** Let $\mathbf{b} \in \mathbb{R}^n$ and let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric and positive definite matrix. Let $0 < \lambda_{\min}$ $\lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_{\max}$ be the n real eigenvalues of $\mathbf{A}$. Prove that the condition number $K_2($ of $\mathbf{A}$ in the 2-norm is $K_2(\mathbf{A}) = \dfrac{\lambda_{\max}}{\lambda_{\min}}$. Suppose that $\mathbf{x}^*$ is an approximation to the solution $\mathbf{Ax} = \mathbf{b}$. Prove that

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|_2}{\|\mathbf{x}\|_2} \leq \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) \frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2}, \quad \text{provide } \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{b} \neq \mathbf{0},$$

where $\mathbf{r}$ is the residual vector for $\mathbf{x}^*$.

3. **(15%)** Let f be sufficiently smooth and satisfy the Lipschitz condition such that there exists unique solution $y(t)$ for $t_0 \leq t \leq t_0 + T$ of the following initial value problem

$$\begin{cases} y'(t) &= f(t, y(t)) \quad \text{for } t_0 < t < t_0 + T, \\ y(t_0) &= y_0 \in \mathbb{R}. \end{cases}$$

A second-order Runge-Kutta method for the numerical approximation of the problem can written as

$$\begin{cases} u_{n+1} &= u_n + h\{a_1 f(t_n, u_n) + a_2 f(t_n + \alpha, u_n + \beta f(t_n, u_n))\}, \\ u_0 &= y_0. \end{cases}$$

Given $a_1 = 1/2$, determine the values of a_2, α and β.

4. **(15%)** Let $\mathbf{b} \in \mathbb{R}^n$ and let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric and positive definite matrix. Define real-valued function $\Phi : \mathbb{R}^n \to \mathbb{R}$ by $\Phi(\mathbf{y}) = \frac{1}{2}\mathbf{y} \cdot \mathbf{Ay} - \mathbf{b} \cdot \mathbf{y}$. Consider the following two problen

$$\text{(L)}: \quad \text{Find } \mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{Ax} = \mathbf{b}.$$

$$\text{(M)}: \quad \text{Find } \mathbf{x} \in \mathbb{R}^n \text{ such that } \Phi(\mathbf{x}) = \min_{\mathbf{y} \in \mathbb{R}^n} \Phi(\mathbf{y}).$$

   (a) Prove that problem (L) is equivalent to problem (M).

   (b) Notice that the gradient of Φ at $\mathbf{y}$ is given by $\nabla\Phi(\mathbf{y}) = \mathbf{Ay} - \mathbf{b}$ and the Hessian of Φ at is given by $D^2\Phi(\mathbf{y}) = \mathbf{A}$ for all $\mathbf{y} \in \mathbb{R}^n$. Derive the gradient method with optimal step si from Taylor's theorem.

5. **(15%)**

   (a) Assume that $u \in C^4([x_0 - h, x_0 + h])$. Use Taylor's theorem and intermediate value theore to derive the following formulas:

$$u'(x_0) = \frac{1}{2h}\Big\{u(x_0+h) - u(x_0-h)\Big\} - \frac{h^2}{6}u'''(\xi) \ \text{ for some } \xi \in (x_0 - h, x_0 + h);$$

$$u''(x_0) = \frac{1}{h^2}\Big\{u(x_0-h) - 2u(x_0) + u(x_0+h)\Big\} - \frac{h^2}{12}u^{(4)}(\eta) \ \text{ for some } \eta \in (x_0 - h, x_0 +$$

   (b) Consider the following two-point boundary value problem:

$$\begin{cases} -\varepsilon u''(x) + \beta u'(x) = 0 & \text{for } \ 0 < x < 1, \\ u(0) = 1 \ \text{ and } \ u(1) = 0, \end{cases}$$

   where ε and β are two positive constants. Let $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ a uniform partition of $[0, 1]$ with mesh size $h > 0$. (i) Formulate the classical second ord centered finite difference scheme for the two-point boundary value problem. (ii) What happe to the solution of part (i) when $\dfrac{\beta h}{2\varepsilon} \gg 1$? (iii) Formulate a finite difference scheme to impro the stability of the centered finite difference scheme.

6. **(15%)** Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with smooth boundary $\partial\Omega$. Consider the followi boundary value problem of reaction-convection-diffusion equation:

$$\begin{cases} -\mu\Delta u + \boldsymbol{\beta}\cdot\nabla u + u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

   where $f \in L^2(\Omega)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)^\top$ is a constant velocity field and $\mu > 0$ is a constant diffusi coefficient.

   (a) Give a variational formulation for the problem using the Sobolev space $V := H_0^1(\Omega)$ and pro the variational problem has a unique solution in V.

   (b) Let $V_h \subseteq V$ be a finite-dimensional finite element space. Formulate the finite element meth for the problem and give an error estimate in the $H^1(\Omega)$-norm with a μ-dependent consta

7. **(15%)** Consider the following initial-boundary value problem of the 1-D heat equation:

$$\begin{cases} u_t &= u_{xx}, \quad t > 0, \ 0 < x < 1, \\ u(x, 0) &= u_0(x), \quad 0 \le x \le 1, \\ u(0, t) &= u(1, t) = 0, \quad t > 0. \end{cases}$$

   (a) Find an explicit finite difference scheme for solving the problem and discuss the stabili properties of the scheme.

   (b) Construct an implicit finite difference scheme to improve the stability of the explicit sche in part (a).

   Hint: the eigenvalues of the $(n-1) \times (n-1)$ tridiagonal matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

   are given by $\mu_i = 4\sin^2\left(\dfrac{i\pi}{2n}\right)$ for $i = 1, 2, \cdots, n-1$.