國立交通大學 應用數學系 博士班資格考試

Discrete Mathematics

2009, September

**1.** (15%) For a given $n$, there is an integer $N(n)$ such that any collection of $N \geqq N($ points in the plane, no three on a line, has a subset of $n$ points forming a conve $n$-gon.

**2.** (15%) Let $G$ be a loopless graph that admits an embedding in the $g$-torus $\mathcal{T}_g$, $g >$

Show that
$$\chi(G) \leq \frac{7+\sqrt{1+48g}}{2}.$$

**3.** (10%) State and prove Dilworth's theorem.

**4.** (10%) State and prove Sperner's theorem.

**5.** (a) (5%) Show that $\sum_{d|n} \phi(d) = n$ and $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1, \\ 0 & \text{otherwise.} \end{cases}$

(b) (5%) State and prove the Möbius inversion formula.

(c) (5%) Let $N_n$ be the number of circular sequences of 0's and 1's, where two sequences obtained by a rotation are considered the same. Show that
$$N_n = \frac{1}{n} \sum_{t|n} \phi\left(\frac{n}{t}\right) 2^t.$$

**6.** (a) (5%) Construct an $OA(4,4)$, i.e. a 4 by 16 matrix $A$ with entries 1,2,3,4, such tha for any two rows of $A$, say row $i$ and row $j$, the 16 pairs $(a_{ik}, a_{jk})$, $1 \leqq k \leqq 16$, are a different.

(b) (10%) Let $m$ and $n$ be positive integers, $m < n$. Show that $m \leqq n/2$ is a necessar and sufficient condition for the existence of a Latin square of order $n$ containing subsquare of order $m$.

**7.** (10%) Let $N$ be the incidence matrix of a symmetric 2-$(v, k, \lambda)$ design. Show tha $N^{\mathrm{T}}$ is also the incidence matrix of a design.

**8.** (10%) Show that a symmetric 2-(29,8,2) design does not exist.