Department of Mathematics

National Chiao Tung University

Ph. D. Qualifying Examination

Fall, 2009.

Analysis

Answer all of the following questions.

1. Let f be a continuous function one-to-one from a compact space X onto a Hausdorff space Y. Prove that f is an homeomorphism.

2. Let (X, μ) be a σ-finite measure space with $\mu(X) = +\infty$. Show that for any finite positive number M, there is some measurable set A in X such that $M < \mu(A) < +\infty$.

3. Suppose a Borel subset A of $(0, 1]$ has measure zero. Let

$$A^2 = \{x^2 : x \in A\}, \quad A^{1/2} = \{\sqrt{x} : x \in A\}, \text{ and } A^{-1} = \{1/x : x \in A\}.$$

Discuss which of A^2, $A^{1/2}$ and A^{-1} have measure zero.

4. Suppose that f is a real-valued differentiable function on $(-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$. Prove that its derivative function f' is Lebesque measurable on $[0, 1]$.

5. Prove that if a real-valued function f is integrable on $[a, b]$ and

$$\int_a^x f(t)\, dt = 0$$

for all x in $[a, b]$ then $f(t) = 0$ a.e. in $[a, b]$.

6. Let $f, f_n \in L^2(\mathbb{R})$ such that $f_n \to f$ a.e., and $\|f_n\| \to \|f\|$. Prove that $f_n \to f$ in norm. Can you obtain a similar result for $L^p(\mathbb{R})$ for arbitrary $1 < p < +\infty$?

7. Show the Dini Theorem: Let f_n be a monotonic decreasing sequence of continuous real-valued functions on a compact space X such that $f_n(x) \downarrow f(x)$ for all x in X. Show that $f_n \to f$ uniformly on X, if f is continuous on X. Give examples to verify the cases X is not compact, or f is not continuous.

8. Let X and Y be compact Hausdorff spaces. Let $C(X)$, $C(Y)$ and $C(X \times Y)$ be the set of all continuous real-valued functions on X, Y, and $X \times Y$, respectively. Show that for each continuous real-valued function f on $X \times Y$ and each $\epsilon > 0$, there are continuous real-valued functions $g_1, \ldots, g_n$ on X and $h_1, \ldots, h_n$ on Y such that for each $(x, y) \in X \times Y$ we have
$$\left| f(x,y) - \sum_{i=1}^{n} g_i(x) h_i(y) \right| < \epsilon.$$

9. Let E, F be Banach spaces and $T : E \to F$ be linear. Equip with E and F the norm topology or the weak topology. Then we can discuss the continuity of T. There are 4 different versions: T is norm-norm continuous, norm-weak continuous, weak-norm continuous, or weak-weak continuous. Show that among these 4 variants, T being norm-norm continuous is the weaker one.

10. Let f be a continuous function on $[0, 1]$. Let P be the class of all polynomials p. Show that a continuous function g in $C[0, 1]$ can be uniformly approximated on $[0, 1]$ by a sequence $p_n(f)$ with $p_n \in P$ if and only if $g(x) = g(y)$ whenever $f(x) = f(y)$.