97.9.19

Qualify Exam: Ordinary Differential Equations

Do **five** problems from the followings, and each problem credits 20 points.

1. Let $f(t,u)$ be a continuous function on a plane rectangle $E:\ t_0 \le t \le t_0+a,\ |u-u_0| \le b$; let $|f(t,u)| \le M$ and $\alpha = \min(a, \frac{b}{M})$. Prove that

$$\begin{cases} u' = f(t,u) \\ u(t_0) = u_0 \end{cases} \tag{1}$$

has a solution $u = u^0(t)$ on $[t_0, t_0+\alpha]$ with the property that every solution $u = u(t)$ of

$$\begin{cases} u' = f(t,u) \\ u(t_0) \le u_0 \end{cases} \tag{2}$$

satisfies $u(t) \le u^0(t)\ \forall t \in [t_0, t_0+\alpha]$.

2. Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.
   (a) Find e^{tA} and the solution $X(t, X_0)$ of

$$\begin{cases} \dfrac{dX(t)}{dt} = AX(t) \\ X(t_0) = X_0. \end{cases} \tag{3}$$

   (b) Find the set of all points $X_0 \in R^3$ such that $\lim\limits_{t\to\infty} ||X(t,X_0)|| = 0$.

3. Consider the equation

$$\ddot{x} + b\dot{x} + 2ax + 3x^2 = 0, \quad b > 0, a > 0. \tag{4}$$

Determine the maximal region of asymptotic stability of the zero solution which can be obtained by using the total energy of the system as a Liapunov function.

4. Consider the van der Pol equation

$$x'' + \epsilon(x^2-1)x' + x = 0,\ \epsilon > 0. \tag{5}$$

   (a) Prove that every solution of (5) is bounded for $t \ge 0$.
   (b) Prove that (5) possesses an asymptotically stable limit cycle for every $\epsilon > 0$. (State the theorem used.)

5. Show that the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 \end{cases} \tag{6}$$

is a Hamiltonian system. Sketch the phase portrait for this system.

6. Suppose $h(x,y)$ is a positive definite function such that $h(x,y) \to \infty$ as $x^2+y^2 \to \infty$. Discuss the behavior in the phase plane of the solutions of the equations

$$\begin{cases} \dot{x} = \epsilon x + y - xh(x,y), \\ \dot{y} = \epsilon y - x - yh(x,y), \end{cases} \tag{7}$$

for all values of ε in $(-\infty, \infty)$.