PDE Qualifying Exam

Feb. 2008

There are five problems in this exam. Please answer each problem and explain your arguments as much detail as possible.

1. (20%) Consider the Schrodinger equation:

$$\frac{1}{i}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad (t,x) \in (0,1) \times (0,\infty).$$

Show that if $u(t,0) = 0$ for all $t \in [\alpha, \beta]$ with $0 < \alpha < \beta < 1$, then $u(t,x) \equiv 0$ in $[\alpha,\beta] \times [0,\infty)$.

2. (20%) Consider the initial value problem for the heat equation $u_t = u_{xx}$, $u(x,0) = u_0(x)$. Suppose that u_0 is real analytic near $x = 0$. Derive a Taylor series expansion for $u(x,t)$ about $x = t = 0$ in terms of that for u_0. Show that even if u_0 is real analytic on $\mathbb{R}$, the resulting series for $u(x,t)$ might not converge for any $(x,t) \neq (0,0)$. Does this contradict the Cauchy-Kowalevski Theorem?

3. (20%) For the equation

$$u = xu_x + yu_y + \frac{1}{2}(u_x^2 + u_y^2)$$

find a solution with $u(x,0) = \frac{1}{2}(1-x^2)$.

4.(a)(10%) Solve the initial value problem: $u_{xx} + 2u_{xt} - 3u_{tt} = 0$ with $u(x,0) = e^x$ and $u_t(x,0) = x^2$.
(b)(10%) Assume that $f(x) = g(x) = 0$ for all $x \in [0,1]$. Let u satisfy the wave equation given in (a) with $u(x,0) = f$ and $u_t(x,0) = g$. Find the largest possible region in (t,x) on which $u(x,t)$ must vanish identically.

5.(20%) Consider the initial-boundary value problem for the wave equation:

$$\begin{cases} \partial_{tt}u - \Delta u + q(x,t)u = f(x,t) \quad (x,t) \in \Omega \times (0,T], \\ u = 0 \quad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = \partial_t u(x,0) = 0 \quad x \in \Omega, \end{cases} \tag{1}$$

where $g(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$. Recall that if $f(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$ then $u(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$. Please derive the following estimates:

$$E(u) := \int_\Omega (|\partial_t u|^2 + |\nabla u|^2 + |u|^2)dx \leq C\|f\|^2_{L^2(\Omega\times[0,T])}$$

and

$$|u(x,t)| \leq C\|f\|^2_{L^2(\Omega\times[0,T])}, \quad \forall\ (x,t) \in \overline{\Omega} \times [0,T],$$

where C is a constant depending on T and Ω. Use these estimates to argue that it is possible to define a (weak) solution of (1) when $f \in L^2(\Omega \times [0,T])$.