# Ph.D. Qualifying Examination on Real Analysis, September 2007

1. (12%) Suppose $f,\, g \in L^1[0,1]$ and $|f(x)g(x)| \geq 1$ for any $x \in [0,1]$. Prove that

$$\left(\int_0^1 |f(x)|\,dx\right)\left(\int_0^1 |g(x)|\,dx\right) \geq 1.$$

2. (12%) Show that the set $\mathcal{K}$ defined by

$$\mathcal{K} = \{f \in C[0,1] : \|f\| \leq 1,\ |f(x) - f(y)| \leq 2|x-y|^{1/2} \text{ for any } x, y \in [0,1]\}$$

is a compact subset of $C[0,1]$. Here $\|\cdot\|$ is the sup norm.

3. (12%) Let $\lambda$ be the Lebesgue measure and let $E$ be a Lebesgue measurable subset of $[0,1]$ such that $\lambda(E \cap [a,b]) \geq \frac{1}{2}(b-a)$ for any $0 \leq a < b \leq 1$. Show that $\lambda(E) = 1$.

4. (12%) Let $A \subset \mathbb{R}$ be a set of Lebesgue measure zero. Show that

$$E = \{(x,y) \in \mathbb{R}^2 :\ x + y \in A\}$$

has 2-dimensional Lebesgue measure zero.

5. (12%) Explain carefully why the following Lebesgue integrals and limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n}{1 + ne^{x^2}}\,dx$$

exist, then quote appropriate theorem(s) to evaluate this limit.

6. (12%) Let $f \in L^1[0,1]$ and $f_k$ be a sequence in $L^2[0,1]$ such that $\|f_k\|_2 \leq 1$ for all $k$. Suppose $f_k \to f$ in $L^1[0,1]$. Show that $f \in L^2[0,1]$ and $\|f\|_2 \leq 1$.

7. (14%) Suppose $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and $F(x) = \int_0^x f(t)\,dt$. Show that for any $x \in \mathbb{R}$, $F$ is not necessarily differentiable at $x$ but

$$\left(\frac{F(x+h) - F(x)}{h}\right) h^{\frac{1}{p}} \to 0 \quad \text{as } h \to 0.$$

8. (14%) Consider the measurable space $(\mathbb{N}, \mathcal{B})$ where $\mathcal{B}$ is the collection of all subsets of $\mathbb{N}$. Let $\delta$ be the Dirac measure at 1 (i.e. $\delta(E) = 1$ if $1 \in E$, $\delta(E) = 0$ otherwise), $\sigma$ be the counting measure, and $\mu$ be the measure defined by

$$\mu(E) \;=\; \sum_{n\in E} \frac{1-(-1)^n}{n^2}, \quad E \in \mathcal{B}.$$

Find pairs of measures from $\delta, \sigma, \mu$ such that one is absolutely continuous with respect to the other. Characterize their Radon-Nikodym derivatives.