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PDE Qualifying Exam (September, 2007)

This exam contains 7 problems with a total of 100 points. Give your arguments as clear as possible.

1. (15 points) This problem tests you the difference between "weak derivative" and "distributional derivative". Consider the Heaviside function $H(x) : U \subset \mathbb{R} \to \mathbb{R}$, $U = (-1, 1)$, given by

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

Clearly $H \in L^1_{loc}(U)$.

   (a) (10 points) Prove or disprove that $H(x)$ has a weak derivative $H'(x)$ on U. In case its weak derivative exists, find it.

   (b) (5 points) Prove or disprove that $H(x)$ has a distributional derivative $H'(x)$ on U. In case its distributional derivative exists, find it.

2. (15 points) Consider the 1-dimensional parabolic Cauchy problem:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = e^{100x^2}, & x \in \mathbb{R}. \end{cases} \tag{0.1}$$

Find an explicit solution (**no integral sign in your solution**) to (0.1). What is the maximal space-time domain of your solution?

3. (20 points) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and assume $u \in C^2(\bar{\Omega})$ satisfies the following

$$\triangle u = f \quad \text{in} \quad \Omega, \qquad u = \varphi \quad \text{on} \quad \partial\Omega$$

where $f \in C^0(\bar{\Omega})$ and $\varphi \in C^0(\partial\Omega)$. Let $B_R(O)$ be the smallest ball centered at the origin O such that $\Omega \subset B_R(O)$. Show that

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\varphi| + \frac{R^2}{2n} \cdot \sup_{\Omega} |f|. \tag{0.2}$$

Also give an example to demonstrate that estimate (0.2) is sharp, i.e., the equality sign can be attained in your example.

Hint: (1). Let $A = \sup_{\partial\Omega} |\varphi|$, $B = \sup_{\Omega} |f|$ and consider the function

$$w(x) = \left[A + \frac{B}{2n}\left(R^2 - |x|^2\right)\right] \pm u(x).$$

(2). For example, take $\Omega = (-R, R)$ to be an interval in $\mathbb{R}$, and consider special f and φ.

4. (10 points) Assume u is a harmonic function on open domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Show that we have the following gradient estimate

$$|Du(y)| \leq \frac{n}{d_y} \cdot \sup_{\Omega} |u|, \qquad d_y = dist(y, \partial\Omega)$$

where $Du(y)$ is the gradient of u at $y \in \Omega$. Hint: You can use the mean value theorem for harmonic functions.

5. (10 points) Assuming that the solution $u(x,y)$ is analytic on $\mathbb{R}^2$, find by **power series expansion** with respect to y the solution of the initial value problem

$$\begin{cases} u_{yy}(x,y) = u_{xx}(x,y) + u(x,y), & (x,y) \in \mathbb{R}^2 \\ u(x,0) = e^x, \quad u_y(x,0) = 0. \end{cases}$$

6. (10 points) Consider the first order quasi-linear equation with initial condition for a function $u(x,y)$ of two variables x, y (we view y as time):

$$\begin{cases} u_y + uu_x = 0 \\ u(x,0) = h(x), \quad x \in \mathbb{R} \end{cases}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a smooth increasing function with $h(5) = 80$, $h'(5) = 3$. Find the values of $u_x(165,2)$ and $u_y(165,2)$.

7. (20 points) The following fact is important in the context of solving wave equations in $\mathbb{R}^n$. Assume $u(y) \in C^\infty(\mathbb{R}^n)$ and define its average on a sphere with center x and radius $r > 0$ as

$$\varphi(x,r) = \frac{1}{n\omega_n r^{n-1}} \int_{|y-x|=r} u(y)\, dS_y$$

where ω_n is the volume of the unit ball $B = B_1(O)$ in $\mathbb{R}^n$. We can rewrite $\varphi(x,r)$ as

$$\varphi(x,r) = \frac{1}{n\omega_n} \int_{|\xi|=1} u(x + r\xi)\, dS_\xi$$

and see that $\varphi(x,r)$ is also defined for $r \in (-\infty, 0)$ with $\varphi(x,-r) = \varphi(x,r)$ for all $r > 0$ and moreover $\varphi(x,r) \in C^\infty(\mathbb{R}^n \times \mathbb{R})$. For fixed x, write $\varphi(r) = \varphi(x,r)$. Show that for any $r > 0$ we have the following identities

$$\varphi'(r) = \frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy \tag{0.3}$$

and

$$\varphi''(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} \Delta u(y)\, dS_y + \frac{1-n}{n} \frac{1}{\omega_n r^n} \int_{B(x,r)} \Delta u(y)\, dy.$$

Note in particular that if $u(y)$ is a harmonic function on $\mathbb{R}^n$, then (0.3) implies the **mean value formula** for harmonic functions.