九十六學年度第一學期博士班資格考試

Number Theory

2007 年 9 月

註： You may quote any standard results without proving them, but state clearly what facts you are assuming. Answers without explanation may receive no credit.

1. (10 %) Let p be an odd prime number and let $\zeta_p = e^{2\pi i/p}$. Show that $\sin(\pi j/p)/\sin(\pi/p)$ is a unit in $\mathbb{Q}(\zeta_p)$ for $1 \leq j \leq p-1$.

2. (10 %) Suppose $c_n \geq 0$ and that

$$\sum_{n \leq x} c_n = Ax + o(x).$$

Show that

$$\sum_{n \leq x} \frac{c_n}{n} = A \log x + o(\log x)$$

as $x \to \infty$.

3. (a) (10 %) Let m be a positive integer and let $S_m \subset \mathbb{C}$ be the subset of the complex numbers consisting of generators of the group μ_m of m-th roots of unity. The m-th cyclotomic polynomial $\Phi_m(X)$ is defined by the following formula

$$\Phi_m(X) = \prod_{\zeta \in S_m} (X - \zeta).$$

Prove that $\Phi_m(X) = \prod_{d|m} \left(X^d - 1\right)^{\mu(m/d)}$ where $\mu(n)$ is the Möbius function.

(b) (10 %) Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function such that the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

converges absolutely for $\Re(s) > s_0$ where s_0 is a positive real number. Show that the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)\, f(n)}{n^s}$$

also converges absolutely for $\Re(s) > s_0$ and that

$$\left[\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right]\left[\sum_{n=1}^{\infty} \frac{\mu(n)\, f(n)}{n^s}\right] = f(1).$$

4. (a) (10 %) Let χ be a character modulo m and let

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{for } \Re(s) > 1$$

be the L-series of χ. Let $L'(s,\chi)$ denote the derivative of $L(s,\chi)$ for $\Re(s) > 1$. Show that

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s} \quad \text{for } \Re(s) > 1$$

where Λ denotes the von Mangoldt function. That is,

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\ell \text{ a power of prime number } p \\ 0 & \text{otherwise.} \end{cases}$$

(b) (10 %) Let K be a number field and let $\zeta_K(s)$ be the Dedekind zeta function of K. For positive integer m, let $\nu(m)$ denote the number of integral ideal J of K with norm $N(J) = m$. Show that if $s > 1$ then

$$\frac{\zeta_K(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

where $\zeta(s)$ denotes the Riemann zeta function and

$$c_n = \sum_{d|n} \mu(d)\nu\left(\frac{n}{d}\right)$$

5. Let $K = \mathbb{Q}(\sqrt{2}, i)$ and $\mathcal{O}_K$ be its ring of integers.

(a) (15 %) Find an integral basis of $K/\mathbb{Q}$ and compute the discriminant of $K/\mathbb{Q}$.

(b) (15 %) Let $U = \mathcal{O}_K^*$ be the group of units in K. Then, as an Abelian group, $U \simeq U_{tor} \times \mathbb{Z}^r$ where U_{tor} denotes the torsion subgroup of U. Determine r and show that U_{tor} is the group of 8^{th}-roots of unity.

(c) (10 %) Determine all prime p in $\mathbb{Z}$ which splits completely in K. Is there any rational prime p which is inert in K? That is, a rational prime number p such that $p\mathcal{O}_K$ is a prime ideal of $\mathcal{O}_K$? Determine all such primes if the answer is yes; otherwise, explain why there doesn't exist rational prime p which is inert in K.