PhD Qualifying Exam for Functional Analysis
September 2007

16年9月

1. (10 pts) Let X be a compact Hausdorff space. We say that a linear functional Λ on $C(X)$ is positive if $\Lambda f \geq 0$ whenever $f \geq 0$. Show that Λ is bounded if Λ is positive.

2. (15pts) Let X be a Banach space and T be an linear operator on X such that that X is the closed span of $\{x, Tx, T^2x, \cdots\}$ for some $x \in X$. Show that if $\dim X < \infty$, then an operator S on X commutes with T if and only if $S = p(T)$ for some polynomial p. Does the conclusion hold if $\dim(X) = \infty$?

3. (15pts) Let X be a Banach space and T be a linear operator on X. Show that T is norm continuous if and only if T is weakly continuous.

4. (10 pts) Let X be a Banach space and T be a bounded linear operator on X with finite spectrum. Show that T can be approximated by invertible operators on X.

5. Let $l^2(\mathbb{N})$ be the Hilbert space of square summable sequences

$$\{(x_1, x_2, \cdots) : \sum |x_n|^2 < \infty\}.$$

We say that S is the unilateral shift on $l^2(\mathbb{N})$ if

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \cdots).$$

Let $\mathcal{B}$ be the algebra of bounded operators on $l^2(\mathbb{N})$.

(1) (5 pts) Let $\phi : \mathcal{B} \to \mathcal{B}$ be defined by

$$\phi(A) = S^*AS, \ A \in \mathcal{B}.$$

Show that ϕ is a bounded linear map on $\mathcal{B}$ and $\|\phi\| = 1$.

(2) (5 pts) Show that ϕ is right invertible, and find a right inverse of ϕ.

(3) (10 pts) Show that $\sigma(\phi)$, the spectrum of ϕ, is the closed unit disc $\overline{\mathbb{D}}$ and, every λ in $\overline{\mathbb{D}}$ is an eigenvalue of ϕ, with eigenspace

$$\ker(\phi - \lambda) = \{A \in \mathcal{B} : a_{i+1,j+1} = \lambda a_{i,j}\},$$

where $(a_{i,j})_{i,j=1}^{\infty}$ is the matrix of A with respect to the standard basis $\mathcal{E} = \{e_n : n = 1, 2, \cdots\}$:

$$e_n = (\delta_{n1}, \delta_{n2}, \cdots),$$

where δ_{ij} is the Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and 0 if else.

(4) (10 pts) Show that $\phi - \lambda$ is right invertible if $|\lambda| < 1$ and find a right inverse of $\phi - \lambda$.

(5) (10 pts) Use (4) to show that $\ker(\phi - \lambda)$ is complemented.

(6) (10 pts) Show that $\mathcal{R}(\phi - \lambda)$ is not closed if $|\lambda| = 1$.