There are six problems in the exam. Work out all six of them.

[20%] 1. State Bruck-Chowla-Ryser theorem and use it to show that a symmetric 2-$(43, 15, 5)$ design does not exist.

[15%] 2. Let $\mathscr{D} = (X, \mathbf{B})$ be a symmetric 2-(v, k, λ) design, and let σ be an automorphism of $\mathscr{D}$. Show that the number of points fixed by σ is the same as the number of blocks fixed by σ.

[20%] 3. Let G be an additive abelian group of order v and let $D_1, \ldots, D_t$ be subsets of size k in G. If the differences arising from the D_i, $1 \leq i \leq t$, give each nonzero element of G exactly λ times, then $D_1, \ldots, D_t$ are said to be a (v, k, λ) difference system in G.

Let $F = (F, +, \cdot)$ be a finite field of order q, and set $F^* = F \setminus \{0\}$. Take a subgroup D of $(F^*, \cdot)$ of order k.

(a) Let $t_1 D, t_2 D, \ldots, t_{(q-1)/k} D$ be the left cosets of D in F^*. Thus, $\cup_{i=1}^{(q-1)/k} t_i D = F^*$ and $t_i D \cap t_j D = \emptyset$ if $i \neq j$. Show that these $t_i D$, $i = 1, \ldots, (q-1)/k$, form a difference system in the additive group $(F, +)$.

(b) What is λ? (Show your arguments.)

[15%] 4. Let $\mathscr{R} = \{0, 1, \ldots, n-1\}$ and let $M_1, M_2, \ldots, M_{n-1}$ be $n-1$ mutually orthogonal latin squares with entries from $\mathscr{R}$. Then there is an affine plane $\mathscr{A}$ of order n such that for any distinct nonzero $s, r \in \mathscr{R}$, the matrices M_s and M_r have the property: for each ordered pair $x, y \in \mathscr{R}$, there is exactly one ordered pair $i, j \in \mathscr{R}$ such that M_s has x in the position (i, j) and M_r has y in the position (i, j). Describe how such an affine plane can be constructed.

[15%] 5. Show that the only latin square of order 4, in normal form, which has an orthogonal mate is:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Find two orthogonal mates of this matrix.

[15%] 6. Let $\mathscr{D}$ be a t design, and let $s < t$ be a positive integer. Show that $\mathscr{D}$ is also an s-design.