# 交通大學 應用數學系 博士班【演算法】資格考 2006, September

## (7 problems in 2 pages)

**1.** (20%) Show that **Bucket sort** algorithm runs in linear expected time under the assumption that the input is generated by a random process that distributes elements uniformly over the interval $[0,1)$.

**2.**(10%) Suppose you are given a set $S=\{a_1,a_2,\cdots,a_n\}$ of tasks, where task $a_i$ requires $p_i$ units of processing time to complete, once it has started. You have one computer on which to run these tasks, and the computer can run only one task at a time. Each task must run non-preemptively, that is, once task $a_i$ is started, it must run continuously for $p_i$ units of time. Let $c_i$ be the ***completion time*** of task $a_i$, that is, the time at which task $a_i$ completes processing. Your goal is to minimize the average completion time $(1/n)\sum_{i=1}^{n} c_i$. Try to give an $O(n\log n)$ algorithm that schedules the tasks so as to minimize the average completion time. Prove that your algorithm minimize the average completion time.

**3**. (10%) Let *T* be a tree. Run **BFS** on any vertex *s* in *T*, remembering the vertex *u* discovered last. Run **BFS** from *u* remembering the vertex *v* discovered last. Show that the distance between *u* and *v* in *T* is the diameter of the tree *T*.

**4.** (20%) Let $G=(V,E)$ be a directed graph with weight function $w:E\to\mathbb{R}$, and let $n=|V|$. We define the ***mean weight*** of a directed cycle $C=\langle e_1,e_2,\cdots,e_k\rangle$ in $G$ to be $\mu(C)=(1/k)\sum_{i=1}^{k} w(e_i)$. Let $\mu^*=\min_C \mu(C)$, where $C$ ranges over all directed cycles in $G$. Assume that every vertex $v\in V$ is reachable from a source vertex $s\in V$. Let $\delta_k(s,v)$ be the minimum weight of all (*s*,*v*)-directed walk of length *k* edges. If there is no directed walk from *s* to *v* with exactly *k* edges, then $\delta_k(s,v)=\infty$.

(a) Show that $\mu^* = \min\limits_{v\in V}\max\limits_{0\le k\le n-1}\dfrac{\delta_n(s,v)-\delta_k(s,v)}{n-k}$.

(b) Give an $O(|V||E|)$-time algorithm to compute $\mu^*$.

**5**. (20%) Give a linear time algorithm that, on input graph $G=(V,E)$, finds a matching with size at least half that of a maximum matching.

**6.** (10%) Try to give an $O(|E|^2 \log|V|)$ algorithm to find a minimum weight cycle in a given edge-weighted, undirected, connected graph $G = (V, E)$ in which all edge weights are non-negative. Prove your algorithm is correct and analyze its running time.

**7.** (10%) The **3-COLOR** problem is "Given a graph $G = (V, E)$, is there a function $c : V \to \{ 1,2,3 \}$ such that $c(u) \neq c(v)$ for every edge $(u, v) \in E$?". Show that **3-COLOR** problem is NP-complete.