DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO-TUNG UNIVERSITY

Ph. D. Qualifying Examination

Fall, 2006.

REAL ANALYSIS

Answer all of the following 7 questions. Each question carries 15 points. The total score is 105 points.

1. Prove that the measure of any countable subset $A = \{a_1, a_2, \ldots\}$ of $\mathbb{R}$ is zero.

2. Prove that if a real-valued function f is integrable on $[a, b]$ and
$$\int_a^x f(t)\,dt = 0$$
for all x in $[a, b]$ then $f(t) = 0$ a.e. in $[a, b]$.

3. Let f be a Riemann integrable function on $[0, 1]$ and $f(x) \geq r > 0$ for all x in $[0, 1]$. Show that
$$\int_0^1 \frac{1}{f(x)}\,dx \geq \frac{1}{\int_0^1 f(x)\,dx}.$$

4. Let f be a real-valued measurable function on a measure space $(X, \mathcal{A}, \mu)$. Show that

   (a) If $\mathcal{B}$ is the class of sets B in $\mathbb{R}$ such that $f^{-1}(B) \in \mathcal{A}$ then $\mathcal{B}$ is a σ–algebra which contains the Borel sets.

   (b) If $\nu(B) := \mu(f^{-1}(B))$ for all B in $\mathcal{B}$ then ν is a measure on $\mathcal{B}$.

5. An extended real valued function $f : \mathbb{R} \longrightarrow [-\infty, +\infty]$ is said to be *lower semicontinuous* at the point y if $f(y) \neq -\infty$ and $f(y) \leq \liminf_{x \to y} f(x)$. Show the following statements.

(a) Let $f(y)$ be finite. Then f is lower semicontinuous at y if and only if given $\epsilon > 0$, there is a $\delta > 0$ such that $f(y) \leq f(x) + \epsilon$ for all x with $|x - y| < \delta$.

(b) A real valued function f is lower semicontinuous on (a, b) if and only if the set $\{x \in \mathbb{R} : f(x) > \lambda\}$ is open for each real number λ.

(c) A lower semicontinuous real valued function f defined on $[a, b]$ bounded from below assumes its minimum on $[a, b]$.

(d) (Dini Theorem) Let $\{f_n\}_n$ be a sequence of lower semicontinuous functions defined on $[a, b]$. Suppose $f_n(x)$ monotonically increasing to 0 for all x in $[a, b]$. Then f_n converges to zero uniformly on $[a, b]$.

6. Let μ_n be a non-decreasing sequence of measures defined on a measurable space $(X, \mathcal{A})$ in the sense that $\mu_n(A) \uparrow \mu(A)$ for all A in $\mathcal{A}$.

(a) Prove that μ is a measure on $(X, \mathcal{A})$ with respect to which all μ_n are absolutely continuous.

(b) On any fixed set A of finite μ measure, let f_n denote the Radon–Nikodym derivative of μ_n with respect to μ. Prove that almost everywhere (with respect to μ, and thus all μ_n) on A, $f_n \uparrow 1$.

7. Suppose we have a real-valued function $K \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} K(x)\, dx = 1$. Let $K_\epsilon(x) = \epsilon^{-n} K(\frac{x}{\epsilon})$. Define

$$f_\epsilon(x) = (f * K_\epsilon)(x) = \int_{\mathbb{R}^n} f(t) K_\epsilon(x - t)\, dt, \quad \forall x \in \mathbb{R}^n.$$

(a) Prove that $f \in L^1(\mathbb{R}^n)$ then $\|f_\epsilon - f\|_{L^1} \to 0$, as $\epsilon \to 0$.

(b) Prove that $C_0^\infty(\mathbb{R}^n)$, the set of all infinitely differentiable functions vanishing at infinity, is norm dense in $L^1(\mathbb{R}^n)$.

(c) Can you prove the same results for $L^p(\mathbb{R}^n)$ with $1 < p < \infty$? How is about the case $p = \infty$?