Qualify Exam: Ordinary Differential Equations
Fall 2006

Do **five** problems from the followings and each problem credits 20 points.

1. Consider the problem $x' = f(t,x), t \geq 0$.
   (a) Show that all solution can be continued on $[0, \infty)$ if $|f(t,x)| \leq kx$.
   (b) Suppose $|f(t,x)| \leq \phi(t)|x|$ and $\int_0^\infty \phi(t)dt < \infty$. Show that every solution approaches to a constant as $t \to \infty$.

2. Suppose $f : R \to R$ is Lipschitz continuous and $g : R \to R$ is continuous. Show that the system $x' = f(x), y' = g(x)y$ has at most one global solution.

3. It is clear that the motion of a simple pendulum could be described by $x'' + k_1 x' + k_2 \sin x = 0$, with $k_1 \geq 0$ and $k_2 > 0$.
   (a) Show that all the solutions are periodic if $k_1 = 0$.
   (b) Discuss the stability of the trivial solution of the given damped system if $|x| < \pi$.

4. Consider the problem $x'' + cx' + rx(1-x) = 0$ with $c, r > 0$.
   (a) By using the phase portrait analysis, discuss the stability of all equilibria.
   (b) Will the problem possesses any periodic solution? Explain your answer.
   (c) Show that the problem has a solution with $0 \leq x \leq 1, x(-\infty) = 1, x(\infty) = 0$ and $x' < 0$ if $c^2 \geq 4r$.

5. Consider the following system
$$x' = \varepsilon x + y - x(x^2 + y^2), \quad y' = -x + \varepsilon y - y(x^2 + y^2).$$
   (a) Show that the equilibrium at the origin is globally asymptotically stable if $\varepsilon \leq 0$.
   (b) Prove that the system possesses the unique limit cycle if $\varepsilon > 0$.

6. Consider a modified prey-predator system
$$x_1' = rx_1(1 - \frac{x_1}{K}) - \frac{\beta x_1 x_2}{\alpha + x_1}, \quad x_2' = sx_2(1 - \frac{x_2}{Lx_1}),$$
   where $r, s, K, L, \alpha, \beta$ are positive. Show that both species will coexist for any initial state $x_1(0) > 0, x_2(0) > 0$.