NCTU PhD Examination, Probability

February, 2006

1.(20 pts) Let X_n be a sequence of independent and identically distributed r.v.'s. Use suitable moment estimates to prove (1) WLLN for X_n in case of finite 2nd moment; (2)SLLN in case of finite 4th moment.

2.(15 pts) Let X have the geometric distribution with success probability $p \in (0, 1)$. Find the expectation, the variance, the probability generating function, and the characteristic function of X.

3.(15 pts) Let Z_λ has a Poisson distribution with mean λ. Use CLT to prove that $(Z_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to $N(0, 1)$, as $\lambda \to \infty$. May consider the case λ be positive integer firstly and then go to the general case.

4.(20 pts) Let X_n be a sequence of r.v.'s and $\mathcal{F}_n$ be an increasing sequence of sub-sigma algebras of $\mathcal{F}$. Assume that each X_n is a submartingale with respect to $\mathcal{F}_n$.

(a) If ϕ is an increasing convex function with $E|\phi(X_n)| < \infty, \forall n$, prove that $\phi(X_n)$ is also a submartingale with respect to $\mathcal{F}_n$.

(b) Prove that $(X_n - a)^+$ is a submartingale if X_n is a submartingale, where a is a real.

5.(15 pts) Let g be a real–valued Lebesgue integrable function on [0,1]. Let U_n be iid uniform on [0,1]. Define $X_i = g(U_i)$. Discuss in what sense does the sample mean $\sum_{i=1}^{n} X_i/n$ aproximate $\int_0^1 g(x)dx$ (Note: This is so–called Monte Carlo).

6. (15 pts) Consider a simple branching model in which we start with one single particle, at time 0. After one unit time, it branches out a random number of subparticles; each subparticle then branches independently and with the same offspring distribution(branching mechanism) as the original one, $\{p_k, k \geq 0\}$. Let Z_n denote the population number of particles at time n. Prove that Z_n is a Markov chain and write down its transition probabilities.