National Chiao Tung University
PhD Qualifying Examination for Graph Theory

※ In this test we follow the definitions and notation in West's book.

1. Show that every minimal nonplanar graph is 2-connected. (10%)
2. Use the Matrix Tree Theorem to show that there are n^{n-2} trees with vertex set $\{1,2,\cdots,n\}$.(10%)
3. Let *G* be a simple *n*-vertex graph, where *n* > 3. Show that if $e(G) > \frac{n^2}{4}$ then *G* contains a triangle. (5%)
4. Show that a graph *G* has a 1-factor if and only if $o(G-S) \le |S|$ for every $S \subseteq V(G)$.(10%)
5. Show that the number of edges in a maximum matching of *G* is $\frac{1}{2}(n(G)-d)$, where $d = \max_{S \subset V(G)} \{o(G-S)-|S|\}$.(10%)
6. Prove that if G is a cubic graph then $\kappa(G) = \kappa'(G)$.(10%)
7. (a) Prove that every planar graph has a vertex of degree at most 5. (5%)
   (b) Prove that every planar graph is 5-colorable. (10%)
8. Prove that if $\kappa(G) \ge \alpha(G)$ then *G* has a Hamiltonian cycle (unless $G=K_2$) (10%)
9. Given an optimal coloring of a *k*-chromatic graph, prove that for each color *s* there is a vertex with color *s* that is adjacent to vertices of the other *k*-1 colors. (10%)
10. Show that if $\binom{n}{p} 2^{1-\binom{p}{2}} < 1$, then $R(p, p) > n$. (10%)