Answer all 7 questions. Total:105 points

Ph.D. Qualifying Examination: Real Analysis

1. (15%) Let $f : [0,\infty) \mapsto [0,\infty)$ be continuous and Lebesgue integrable on $[0,\infty)$.

   (a) Prove that $x\left(\inf_{t\geq x} |f(t)|\right) \to 0$ as $x \to \infty$.

   (b) Can you conclude $\lim_{x\to\infty} f(x) = 0$? Justify your answer.

2. (15%) Find the value of the following limit:

$$\lim_{n\to\infty} \int_0^{\pi} \left(1 - \frac{x}{n}\right)^n \sin x \, dx.$$

3. (15%) Let $A = (a_{j,k})$ be an $n \times n$ matrix with $a_{j,k} \geq 0$ for all j, k. For $X = (x_1, x_2, \cdots, x_n)$, let AX denote the vector $Y = (y_1, y_2, \cdots, y_n)$ with

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

   (a) Set $1/p + 1/p^* = 1$, where $1 < p < \infty$. Prove that

$$\sup_{X\neq 0} \frac{\|AX\|_1}{\|X\|_p} \leq \left\{\sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{j,k}\right)^{p^*}\right\}^{1/p^*}.$$

   (b) Can we replace "$\leq$" in (a) by "$=$"? Justify your answer.

4. (15%) Let $f_n \in L^2([-\pi,\pi])$ with $\|f_n - f_{n+1}\|_2 \leq 1/n^2$ for all $n \geq 1$. Prove that $\sum_{n=1}^{\infty} |f_n(x) - f_{n+1}(x)| < \infty$ almost everywhere on $[-\pi,\pi]$ and $\{f_n\}_{n=1}^{\infty}$ converges in $L^2([-\pi,\pi])$.

5. (15%) Let $f : [a,b] \mapsto \mathbb{R}$ be Lebesgue measurable. Set

$$E_\alpha = \{x \in [a,b] : |f(x)| > \alpha\}$$

   and $\omega(\alpha)$ is the Lebesgue measure of E_α. Prove that $\omega : [0,\infty) \mapsto \mathbb{R}$ is Borel measurable and

$$\int_a^b |f(x)|^p dx = p\int_0^{\infty} \alpha^{p-1}\omega(\alpha) d\alpha \qquad (1 \leq p < \infty).$$

6. (15%) Let $C[0,1]$ denote the space of all real-valued continuous functions defined on $[0,1]$. Assume that $f \in C[0,1]$ and

$$\int_0^1 x^n f(x)\,dx = 0 \qquad \text{for all } n = 0, 1, 2, \cdots.$$

(a) Set $\Omega = \{g \in C[0,1] : \int_0^1 g(x)f(x)\,dx = 0\}$. Prove that Ω is a dense subset of $C[0,1]$.

(b) Prove that $f = 0$ on $[0,1]$.

7. (15%) Let $1 \le p < \infty$ and T be a bounded linear functional on $L^p(\mathbb{R})$. Set $\Phi(s) = T(\chi_{[0,s]})$, where $\chi_{[0,s]}$ denotes the characteristic function of the interval $[0,s]$. Prove that Φ is absolutely continuous on $[0,1]$.