# PDE Qualifying Exam.

September, 2005

\# 1. (20 pts) Consider two PDE problems given by

$$\text{(A)} \quad u_t = u_{xx}, \quad x \in \mathbb{R}, t > 0\,,$$

$$\text{(B)} \quad u_t = u \times u_x, \quad x \in \mathbb{R}, t > 0\,,$$

with initial data $u|_{t=0} = u_0(x)$. Suppose $u_0 \in C^1(\mathbb{R}) \bigcap H^1(\mathbb{R})$. Can (A) and (B) have the solution $u(t, \cdot) \in C^1(\mathbb{R})$ for $t > 0$? Prove or disprove your answer.

\# 2. (20 pts) Consider two PDE problems given by

$$\text{(C)} \quad u_{tt} = \Delta u, \quad x \in \mathbb{R}^n, n = 2, t > 0\,,$$

$$\text{(D)} \quad u_{tt} = \Delta u, \quad x \in \mathbb{R}^n, n = 3, t > 0\,,$$

where $\Delta \equiv \sum_{j=1}^{n} \partial_{x_j}^2$ is the standard Laplace operator.

(i) State suitable initial data of (C) and (D) such that they are well-posed.

(ii) What's the difference between the solutions of (C) and (D)?

\# 3. (20 pts) Let $v_0 \in C^\infty(\mathbb{R})$ be a smooth and periodic function with period $2\pi$. Can there exist another function $v = v(t, x)$ satisfying

(i) $v(0, x) = v_0(x), \quad \forall x \in \mathbb{R}\,,$

(ii) $\forall t > 0, \quad v(t, \cdot)$ is periodic with period $2\pi$,

(iii) $v_t = v_{xx}, \quad \forall x \in \mathbb{R}\,, t > 0\,?$

Prove or disprove your answer.

\# 4. (20 pts) Use "separation of variable" to solve the boundary-valued problem given by

$$\begin{cases} i u_t &= u_{xx}\,, \quad x \in [0, 2\pi], t > 0\,, \\ u &= 0\,, \qquad x = 0\,, 2\pi\,, t > 0\,, \end{cases}$$

where $i = \sqrt{-1}$ and $u = u(t, x) \in \mathbb{C}$.

\# 5. (20 pts) Assume $u \in C^2(\mathbb{R}^n \backslash \{0\}), n \geq 3$ and $u \geq 0$ is harmonic in $\mathbb{R}^n \backslash \{0\}$. Show that $u$ has the form that

$$u(x) = \frac{C_1}{|x|^{n-2}} + C_2\,, \forall x \in \mathbb{R}^n \backslash \{0\}\,,$$

for some constants $C_1, C_2 \geq 0$.