PhD Qualifying Examination for Discrete Mathematics
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In this test we follow the definitions and notation in West's book. For instance, in a graph we allow multiple edges and loops.

1. **(15%)** The *odd graph* O_k is the graph whose vertices are the k-element subsets of $\{1, 2, \ldots, 2k+1\}$, and two vertices are adjacent if and only if they are disjoint. Prove that the girth of O_k is 6 if $k \geq 3$. Recall that the *girth* of a graph is the minimum length of a cycle in this graph.

2. **(15%)** Compute the number of spanning trees of the complete bipartite graph $K_{3,m}$.

3. **(14%)** Let G be a k-regular graph of even order that remains connected when any $k-2$ edges are deleted. Prove that G has a 1-factor.

4. **(14%)** A *cactus* is a connected graph in which every block is an edge or a cycle. Prove that the maximum number of edges in a simple n-vertex cactus is $\lfloor 3(n-1)/2 \rfloor$.

5. **(14%)** Prove that a chordal graph has at least one simplicial vertex. Recall that a graph is *choral* if it does not contain any induced cycle of length greater than three; a vertex is *simplicial* if all the vertices adjacent to it form a clique.

6. **(14%)** Let G be an n-vertex simple planar graph with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges. Use this to prove that the Petersen graph is non-planar. Notice that the Petersen graph is isomorphic to the odd graph O_2.

7. **(14%)** Use Tutte's 1-factor Theorem to prove that every connected line graph of even order has a perfect matching. Conclude from this that the edges of a simple connected graph of even size can be partitioned into paths of length 2.