93年9月
常微 資格考

1. (20%) Let t be a real scalar, D be an open set in R^{n+1} with an element of D written as (t,x), and $f : D \to R^n$ be a continuous function. Consider the following differential equation

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases} \tag{1}$$

Prove the following statements respectively :
(a) For any $(t_0, x_0) \in D$ there is at least one solution of (1) passing through (t_0, x_0).
(b) If, in addition, $f(t,x)$ is locally lipschotzian with respect to x in D, then for any (t_0, x_0) in D, there exists a unique solution $x(t, t_0, x_0)$ of (1) passing through (t_0, x_0). Furthermore, the domain E in R^{n+2} of definition of the function $x(t, t_0, x_0)$ is open and $x(t, t_0, x_0)$ is continuous in E.

2. (20%) Consider the van der Pol equation

$$x'' + \epsilon(x^2 - 1)x' + x = 0. \tag{2}$$

Prove the following statements.
(a) Equation (2) has a unique asymptotically stable limit cycle Γ for every $\epsilon > 0$. (State the theorem used.)
(b) Let $D = \{(x,y) | x^2 + y^2 < 3\}$. Then $\Gamma \not\subset D$. (State the theorem used.)

3. Consider the following Predator-Prey system

$$\begin{cases} \dfrac{dx}{dt} = x(\gamma(1 - \dfrac{x}{K}) - \dfrac{my}{a+x}) \\ \dfrac{dy}{dt} = (\dfrac{mx}{a+x} - d)y, \qquad \gamma, K, m, a, d > 0 \\ x(0) > 0, y(0) > 0. \end{cases} \tag{3}$$

For various possible cases, do the following :
(a) (15%) Do the stability analysis for each equilibrium with nonnegative components.
(b) (5%) Find the stable manifold of each saddle point.

4. (20%) If $p(t+\pi) = p(t) \not\equiv 0$ for all t, $\int_0^\pi p(t) \geq 0$, $p(t)$ is continuous, real and

$$\int_0^\pi |p(t)| dt \leq \frac{4}{\pi},$$

then prove that all solutions of the equation

$$x'' + p(t)x = 0, \tag{4}$$

are bounded on $(-\infty, \infty)$.

5. (20%) Suppose $g(x,y)$ has continuous first derivatives with respect to x, y. Show there is an $\epsilon_0 > 0$ such that for $|\epsilon| < \epsilon_0$ the equation

$$x'' - k(1 - x^2)x' + x = \epsilon g(x, x'), \; k > 0 \tag{5}$$

has a unique periodic solution in a neighborhood of the unique periodic orbit of the van der Pol equation

$$x'' - k(1 - x^2)x' + x = 0. \tag{6}$$