Qualify Examination
Design theory

1. A **t-(v,k, λ) design** is an ordered pair (**X**,**B**) where |**X**|=v, **B**=$\{B_1,B_2,\ldots,B_n\}$ is a collection of k-subsets (blocks) of **X**, such that any t-subset of **X** appears in exactly λ blocks. The **block intersection number** $\lambda_{i,j}$ is the number of blocks B_k such that $B_k \cap \{1,\ldots,i\} = \{1,\ldots,j\}$ for $j \leqq i$. Prove or disprove the existence of the following t-(v,k, λ) designs: (30%)
   a. 2-(7,3,1),
   b. 3-(8,4,1),
   c. 4-(9,5,1),
   d. 2-(7,4,2),
   e. 2-(22,7,2),
   f. 3-(10,4,1).

2. Let $\mathbf{N_{11}}$ be the set of integers modulo 11,
   $B_0 = \{1, 3, 4, 5, 9\}$, $B_i = (B_0 + i) \bmod(11)$, $1 \leqq i \leqq 10$, $C_i = \mathbf{N_{11}} - B_i$, $C_{i,j} = B_i \cup B_j - B_i \cap B_j$, $B_{i,j} = \mathbf{N_{11}} - C_{i,j}$, $0 \leqq j < i \leqq 10$. (30%)
   a. Prove that 11 blocks of 5-subsets $\{B_0,\ldots,B_{10}\}$ form a 2-(11,5,2) design.
   b. Prove that 66 blocks of 5-subsets $\{B_0,\ldots,B_{10}, B_{i,j}, 0 \leqq j < i \leqq 10\}$ form a 4-(11,5,1) design and find all $\lambda_{i,j}$.
   c. Let $\mathbf{N} = \mathbf{N_{11}} \cup \{*\}$, where * is an element not in $\mathbf{N_{11}}$, $A_i = B_i \cup \{*\}$, $A_{i,j} = \{*\} \cup B_{i,j}$. Prove that 132 blocks of 6-subsets $\{A_i, C_i, A_{i,j}, C_{i,j}, 0 \leqq j < i \leqq 10\}$ form a 5-(12,6,1) design and find $\lambda_{i,j}$ for all $0 \leqq j \leqq i \leqq 5$.
   d. The 5-design constructed above is called the **small Witt design**. The well-known 5-(24,8,1) design is called **large Witt design**. Please find all $\lambda_{i,j}$ for large Witt design.
   e. Prove or disprove the size of the intersection of any two blocks of the small Witt design is 0, 2 or 4.
   f. Try to construct 3-(22,6,1), 4-(23,7,1) designs.

3. Let $\mathbf{B} = \{B_1, B_2, \ldots, B_n\}$ be a collection of subsets of a finite set of size v. B is called a **2-disjunct system of order v** if $B_i \cup B_j$ is not contained in B_k for all distinct indices i,j,k. (20%)

    a. Prove the size of a 2-disjunct system of order v is at least v(v-1)/6. (hint: consider a Steiner triple system).
    b. Try to construct a 2-disjunct system of order 23 with size larger than 200.

4. Prove that you can explicitly construct all **Hadamard** matrices of order $\leqq 80$. (20%)

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