PDE Qualifying Examination (Feb. 2004)

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Answer ALL of the following problems. Each problem carries 20 % .

1. Find all the solutions to the following PDE problems

   (a) $(y-z)u_x + (z-x)u_y + (x-y)u_z = 0$ with initial curve $x = t,\ y = 3t,\ z = 0\ (t \in \mathbf{R})$;

   (b) $u_{tt} - 9u_{xx} = 6x$ for $t > 0, x \in \mathbf{R}$ such that $u(x,0) = 1,\ u_t(x,0) = 1$ for $x \in \mathbf{R}$.

2. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$Lu = \Delta u + \sum_{k=1}^{n} a_k(x) u_{x_k} + c(x)u = 0 \ ,$$

   where $c(x) < 0$ in Ω, and Ω is a bounded domain in $\mathbf{R}^n$.

   (a) Show that $u = 0$ on $\partial\Omega$ implies $u = 0$ in Ω.

   (b) Hence show that the solution of the Dirichlet problem

$$Lu = f \text{ in } \Omega, \qquad u = g \text{ in } \partial\Omega \ ,$$

   for continuous f and g, if exists, is unique.

3. In 3 dimensions, let $Lu = \Delta u + cu$, where c is a real constant.

   (a) Show that the radially symmetric solutions of the equation $Lu = 0$ are

$$u(r) = \frac{1}{r}(c_1 \cos\sqrt{c}r + c_2 \sin\sqrt{c}r)$$

   (b) Show that the function

$$K(x,\xi) = -\frac{\cos(\sqrt{c}r)}{4\pi r}, \qquad r = |x - \xi| \ ,$$

   is a fundamental solution for L with pole ξ.

4. Consider the Cauchy problem for the heat equation

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } x \in \mathbf{R},\ t > 0; \\ u(x,0) &= f(x) \quad \text{for } x \in \mathbf{R},\ t = 0, \end{aligned}$$

where f is bounded and continuous.

(a) If $\int_{\mathbf{R}} |f(y)|\, dy < \infty$, show that there exists $C > 0$ such that

$$|u(x,t)| \leq C\, t^{-1/2} ,$$

so that $\lim_{t\to\infty} u(x,t) = 0$.

(b) If $\int_{\mathbf{R}} |f(y)|^p\, dy < \infty$ $(p > 1)$. Can you find similar estimates as above?

5. (a) Let $u \not\equiv 0$ satisfy $u \in C^2(\mathbf{R}^n)$, $\Delta u = 0$ on $\mathbf{R}^n$. Show that $u \notin L^2(\mathbf{R}^n)$.
Hint: You may use mean value property or otherwise.

(b) Let $\Omega \subset \mathbf{R}^n$ be a bounded, open set with a smooth boundary $\partial\Omega$, and $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \overline{\Omega}_T - \Omega_T$, where $T > 0$. Suppose $u \in C^2(\overline{\Omega}_T)$ is a solution of the problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma_T, \\ u_t = 0 & \text{on } \Omega \times \{t = 0\}. \end{cases}$$

Prove that $u = 0$ on Ω_T.
Hint: Define $E(t) = \int_\Omega u_t^2 + |\nabla u|^2 dx$ $(0 \leq t \leq T)$.

End of Paper