博士班資格考 一數值分析

92月2月

1. (20%) To solve the two-points boundary value problem

$$\begin{cases} -u''(x) = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1a}$$

one may apply the finite difference method by replacing $-u''(x) = f(x)$ by

$$\frac{-U_{i-1} + 2U_i - U_{i+1}}{h^2} = f_i, \quad 1 \le i \le m, \tag{1b}$$

at $x_i = ih$, where $h = \frac{1}{m+1}$ and $U_0 = U_{m+1} = 0$.

(a) Discuss the local truncation error, global error, stability, consistency and convergence of (1).

(b) Prove that (1b) converges (of order two) with respect to $\| \ \|_\infty$-norm.

Hint: $\|A^{-1}\|_\infty = \|A^{-1}e\|_\infty$, where $e = [1, \ldots, 1]^\top$ and $e(x) \equiv \frac{x(1-x)}{2}$ is the solution of $\begin{cases} -u''(x) = 1, \\ u(0) = u(1) = 0. \end{cases}$

2. (20%) Let $\{\phi_n(x) | n \ge 0\}$ be an orthogonal family of polynomials on (a, b) with weight function $w(x) \ge 0$. Given any $f(x) \in C[a, b]$ define $\|f\|_2^2 = \int_a^b w(x) f^2(x)\, dx$.

(a) Prove that the polynomial $\phi_n(x)$ has exactly n distinct real roots in (a, b).

(b) Show that there is uniquely the best polynomial $r_n^*(x)$ of degree n to approximate $f(x)$ by means of the least squared approximation with the given weight function $w(x)$.

3. (15%) Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $f \in C^{2n+2}[a, b]$ which is interpolated by a polynomial $p(x)$ of degree $2n + 1$ satisfying

$$\begin{cases} p(x_i) &= f(x_i), \\ p'(x_i) &= f'(x_i), \quad i = 0, 1, \ldots, n. \end{cases} \tag{2}$$

Show that for the error at a point $x \in [a, b]$ holds: there exists $\xi \in [a, b]$ such that

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} [w_{n+1}(x)]^2,$$

where $w_{n+1}(x) = \prod_{k=0}^{n} (x - x_k)$.

P2

4. (20%)

(a) For the trapezoidal rule denoted by $I_n^{(T)}$ for evaluating $I = \int_b^a f(x)\, dx$ we have the asymptotic error formula

$$I - I_n^{(T)} = -\frac{h^2}{12}[f'(b) - f'(a)] + O(h^4)$$

and for the midpoint formula $I_n^{(M)}$, we have

$$I - I_n^{(M)} = \frac{h^2}{24}[f'(b) - f'(a)] + O(h^4).$$

Using these results obtain a new numerical formula $\widetilde{I}_n$ combining $I_n^{(T)}$ and $I_n^{(M)}$ with a higher order of convergence, write out the weights to the new formula $\widetilde{I}_n$.

(b) State the Gaussian-Legendre quadrature formula for the integration $\int_{-1}^1 f(x)\, dx$ and list continuous function which give no error in evaluating the integral on $[-1, 1]$ by the above Gaussian quadrature formula.

5. (20%) For the IVP $y' = f(x, y),\ y(x_0) = y_0$.

(a) Give the Euler method (explicit one step method) and the trapezoidal method (implicit one step method).

(b) Use the Euler method as a predictor and the trapezoidal method as a corrector to develop a Predictor-Corrector algorithm.

6. (20%) Consider the linear system $Ax = b$ with A symmetric positive definite. Show that the Conjugate-Gradient method yields the solution after q steps, if either A has only q different eigenvalues or $x_0 = 0$ and b lies in a q-dimensional invariant subspace of A. Treat the case $A = \mathrm{I} + ww^\top$ $(w \neq 0)$ explicitly.

Hint: Conjugate-Gradient method
Given x_0, $r_0 = b - Ax_0 = p_0$, $k = 0$
while $r_k \neq 0$
$\quad \alpha_k \quad = r_k^\top r_k / p_k^\top A p_k$
$\quad x_{k+1} = x_k + \alpha_k p_k$
$\quad r_{k+1} = b - x_{k+1} = r_k - \alpha_k A p_k$
end
$\quad x = x_k$.