博士班資格考

P1

Algorithms

92年2月

Do any 5 problems.

1. (20%) Give asymptotic lower and upper bounds for $T(n)$, $n > 0$, in each of the following recurrences. Make your bound as tight as possible. Justify your answers.
   (a) $T(1) = 0$, $T(n) = T(\lceil 9n/10 \rceil) + n$, for $n > 1$.
   (b) $T(1) = 0$, $T(n) = T(\lfloor \sqrt{n} \rfloor) + 1$, for $n > 1$.

2. (20%) The subgraph isomorphism problem is to determine whether an undirected graph contains a subgraph which is isomorphic to another undirected graph. The satisfiability problem is to determine whether a Boolean formula has a truth assignment to satisfy the formula. The clique problem is to determine whether a graph contains a complete subgraph of size k. The Hamiltonian problem is to determine whether an undirected graph contains a Hamiltonian cycle. Assume that the satisfiability problem, the clique problem and the Hamiltonian problem are all known to be NP-complete. Prove that the subgraph isomorphism problem is NP-complete.

3. (20%) Let $L_1 = y_1, y_2, \ldots, y_m$ and $L_2 = z_1, z_2, \ldots, z_n$ be two sorted lists of integers. Given an integer x, design an $O(m+n)$ time algorithm to determine if there is an integer y in L_1 and an integer z in L_2 such that $x = y + z$.

4. (20%) Let $X = x_1, x_2, \ldots, x_n$ be a sequence of n numbers. A subsequence of X is a sequence obtained from X by deleting some elements. Design an $O(n^2)$ algorithm to find the longest monotonically increasing subsequence of a given sequence X.

5. (20%) Given a weighted connected undirected graph G with vertex set V. Assume that the weight of each edge is positive, and no two edges have the same weight. For any vertex $v \in V$, define e_v to be the edge with minimum weight among all edges incident at v. Let F be the subset of edges defined by $F = \{e_v | v \in V\}$.
   (a) For every vertex $v \in V$, show that there is a minimum spanning tree T of G such that T contains the edge e_v.
   (b) Show that there is a minimum spanning tree T of G such that T contains the edge set F.

(c) Prove or give a counter-example that the induced subgraph $G[F]$ is a minimum spanning tree of G.

6. (20%) Let $a_1, a_2, \ldots, a_n$ be a sequence of n distinct integers. If $i < j$ and $a_i > a_j$, then the pair(i, j) is called an *inversion* of the sequence.

   (a) Briefly state the *insertion sort algorithm*, and state the relationship between the running time of the insertion sort algorithm and the the number of inversions in the input sequence.

   (b) Give an algorithm to determine the number of inversions in the input sequence of n distinct integers in $O(n \log n)$ time. Hint: Modify the merge sort algorithm.

7. (20%) Quick-sort is a sorting algorithm which can be described as follows. It chooses a value x and partitions the array to be sorted into two parts. The first part contains elements less than or equal to x and the second part contains elements greater than or equal to x. It then sorts both parts recursively. A method to implement quick-sort non-recursively is to use a stack to store the ranges of the array yet to be sorted. The algorithm repeatedly sorts those part of the array whose range is stored on the top of the stack until the stack is empty. Show that if the algorithm stores the smaller part on top of the stack, then the maximum height of the stack is bounded by $O(\log_2 n)$.