Department of Applied Mathematics National Chiao Tung University

Ph.D. Qualifying Examination    Analysis    Spring, 2018

1. Let $(X, \mathcal{B}, \mu)$ be a measure space.
(a)(10%) For any $p > 0$, let $f, f_1, f_2, \cdots \in L^p$ such that $f_n \to f$ a.e. Then $f_n \to f$ in $L^p$ if and only if $\|f_n\|_p \to \|f\|_p$.
(b)(10%) As in (a) and $f_n \to f$ in $L^1$. Then, for any set $A \in \mathcal{B}$, $\int_A |f_n| d\mu \to \int_A |f| d\mu$.

2.(20%) Let $\Omega$ be a bounded domain of $\mathbb{R}^n$. Let $f$ be an essentially bounded function on $\Omega$ (with respect to the Lebesgue measure) with $\|f\|_{L^\infty(\Omega)} > 0$. Denote

$$a_n = \int_\Omega |f|^n dx, \quad n = 1, 2, \cdots .$$

Show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_{L^\infty(\Omega)}.$$

3.(20%) Let $\omega_n$ be the volume of the unit sphere in $\mathbb{R}^n$. Show by the Fubini theorem that

$$\omega_n = 2\omega_{n-1} \int_0^{\pi/2} \cos^n \theta d\theta,$$

$n = 2, 3, \cdots$. Let $v_n(r)$ be the volume of the sphere with radius $0 < r < 1$ in $\mathbb{R}^n$. Find the limit

$$\lim_{n\to\infty} \frac{v_n(r)}{\omega_n}.$$

4.(20%) Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a linear operator and $Tf \geq 0$ whenever $f \geq 0$.
(a) Show that $\|T(|f|)\| \geq \|Tf\|$ for all $f \in L^2(\mathbb{R})$.
(b) Show that $T$ is bounded.

5. Let $\mathcal{S}$ be the space of measurable, finite-valued a.e. functions on the measure space $(X, \mathcal{B}, \mu)$ with $\mu(X) < \infty$. For $f, g \in \mathcal{S}$, define

$$d(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} d\mu.$$

Show that
(a)(10%) for $f_n \in \mathcal{S}$, $f_n \to 0$ in measure if and only if $d(f_n, 0) \to 0$.
(b)(10%) $\mathcal{S}$ is a complete metric space with respect to $d$.