105 學年度第一學期博士班資格考

偏微分方程

1. (10%) Find a solution of

$$u_{tt} - c^2 u_{xx} = \lambda^2 u$$

of the form $u = f(x^2 - c^2 t^2)$, where λ is a nonzero constant and $f(0) = 1$.

2. (20%) Assume Ω is connected. Use (a) the energy method and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

are $u \equiv \text{constant}$.

3. (20%) Assume $u \in C^2(\mathbb{R}^n \setminus \{0\}), n \geq 3$ and $u \geq 0$ is harmonic in $\mathbb{R}^n \setminus \{0\}$. Show that u has the form that

$$u(x) = \frac{C_1}{|x|^{n-2}} + C_2, \forall x \in \mathbb{R}^n \setminus \{0\},$$

for some constants $C_1, C_2 \geq 0$.

4. (20%) Let u be the solution of the following initial value problem of heat equation:

$$\begin{cases} u_t = \Delta u & \text{for } x \in \Omega, t > 0, \\ u = 0, & \text{on } x \in \partial\Omega, t > 0, \\ u = \delta_0 & \text{at } t = 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain with $0 \in \Omega$ and δ_0 is the standard Dirac measure concentrating at the origin.

Let G be the standard heat kernel on $\mathbb{R}^n \times (0,\infty)$

Define $w = u - G$.

Which of the following statements is (are) true?

(A) $\sup_{x\in\Omega, t>0} |w(x,t)| < \infty$

(B) $\sup_{x\in\Omega, t>0} |w(x,t)| = \infty$

(C) w is smooth on $\Omega \times (0,\infty)$

(D) There exists $x_0 \in \Omega$ such that

$$|w(x,t)| \to \infty \text{ as } (x,t) \to (x_0, 0)$$

Find and justify your answer.

5. (20%) Let $u \in C^2(\mathbb{R}\times[0,\infty))$ solve the initial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} = u_{xx} & \text{in } \mathbb{R}\times(0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R}\times\{t=0\}. \end{cases}$$

Suppose g, h have compact support. Let

$$k(t) = \int_{\mathbb{R}} u_t^2 \, dx$$

and

$$p(t) = \int_{\mathbb{R}} u_x^2 \, dx$$

for $t \geq 0$. Which of the following statements is (are) true?

(A) $\frac{d}{dt}(k(t) + p(t)) = 0$ for $t > 0$

(B) $\frac{d}{dt}(k(t) + p(t)) \neq 0$ for $t > 0$

(C) $\lim_{t \to \infty}(k(t) - p(t)) = 0$

(D) $\lim_{t \to \infty}(k(t) - p(t)) \neq 0$

Find and justify your answer.

6. (10%) Show that the solution u of the quasi-linear partial differential equation $u_y + a(u)u_x = 0$ with initial condition $u(x, 0) = h(x)$ is given implicitly by $u = h\left(x - a\left(u\right)y\right)$.

Suppose $a(h(s))$ is a strictly decreasing function of s.

Which of the following statements is true?

(A) The solution u is regular for all $y > 0$,

(B) The solution u becomes singular for some $y > 0$.

Find and justify your answer.