NCTU Department of Applied Mathematics

Discrete Mathematics Qualifying Examination, September 2016

• Note: Show your work, as partial credits will be given. You will be graded not only on the correctness of your answers, but also on the clarity with which your express it. Be neat.

Problem 1. (10=(2×5) pts) State the following theorems (proofs are not required).
(1) Erdös-Ko-Rado theorem (on sets).
(2) Hall's marriage theorem (on bipartite graphs).
(3) König's theorm (on $(0, 1)$-matrices).
(4) Dilworth's theorem (on posets).
(5) Stanley's theorem (on acyclic orientations).

Problem 2. (15 pts) A k-element subset (k-set for short) of $[n] := \{1, 2, \ldots, n\}$ is *even* if the sum of its elements is even. Otherwise it is an *odd* set. For example, For $n = 5$, the 2-set $\{1, 3\}$ is even, while the 2-set $\{1, 4\}$ is odd. Let $a_{n,k}$ be the difference of the number of even k-sets and the number of odd k-sets of $[n]$. Find the formula of $a_{n,k}$.

Problem 3. (15 pts) Prove that if a simple graph on n vertices has e edges, then it has at least

$$\frac{e}{3n}(4e - n^2)$$

triangles.

Problem 4. (15 pts) For a 2-(v, k, λ) design with b blocks and $v > k$, prove that

$$b \geq v.$$

(A t-(v, k, λ) is a collection of k-subsets (called 'blocks') over the ground set V of size v, and for any set $T \subset V$ of t elements there are exactly λ blocks contain all elements in T.)

Problem 5. (15 pts) Count the number N_n of bicolor necklaces of size n. (A bicolor necklace of size n is a circular sequence of 0's and 1's of n letters, where two sequences obtained by a rotation are considered the same).

Problem 6. (15=(8+7) pts) Let $s_{n,k}$ be the (signed) Stirling number of the first kind (i.e. $s_{n,k} = (-1)^{n-k} c_{n,k}$, where $c_{n,k}$ is the number of permutations in $\mathfrak{S}_n$ of k cycles in the cycle dicomposisions), and $S_{n,k}$ be the Stirling number of the second kind (i.e. $S_{n,k}$ is the number of partitions of an n-set into k nonempty sets).
(1) Given $k \geq 0$. Find the generating function $\sum_{n \geq k} s_{n,k} \frac{z^n}{n!}$.
(2) Given $k \geq 0$. Find the generating function $\sum_{n \geq k} S_{n,k} \frac{z^n}{n!}$.

Problem 7. (15=(5+10) pts) Let $G = (V, E)$ be a simple graph with its chromatic polynomial

$$\chi_G(\lambda) = \sum_{i \geq 0} a_i \lambda^{n-i}.$$

(1) Prove that $a_1 = -|E|$.
(2) Find an interpretation for a_2.