105 學年度第一學期博士班資格考

分　析

1. Let $(X, \mathcal{B}, \mu)$ be a measure space. Assume that $\{f_n\}$ is a sequence of measurable functions which are finite a.e. Prove or disprove the following statements:

(a) (10%) If $f_n \to f$ in measure, then $f_n \to f$ in L^1 norm.

(b) (10%) If $\mu(X) < \infty$ and $f_n \to f$ a.e. (pointwise), then $f_n \to f$ in measure.

(c) (10%) The restriction $\mu(X) < \infty$ of (b) is not necessary. Namely, for any $(X, \mathcal{B}, \mu)$, $f_n \to f$ a.e. implies $f_n \to f$ in measure.

2. Let H be a Hilbert space with a complete orthonormal basis $\{e_j\}$.

(a) (10%) Assume that $\{f_k\}$ is a bounded sequence in H and $f \in H$. If, for any j, $(f_k, e_j) \to (f, e_j)$ as $k \to \infty$, then $f_k \to f$ weakly, where $(\cdot, \cdot)$ is the inner product in H.

(b) (10%) Does the statement (a) remain true for any sequence (may be unbounded) $\{f_k\}$ in H?

3.(10%) Let F be a nonempty closed subset of $(0,1)$. For $x \in (0,1)$, define $d(x) = \text{dist}(x, F)$, the distance of x to F. Let

$$M(x; F) = \int_0^1 [-\log d(y)]^{-1} |x - y|^{-1} dy,$$

prove that for $x \in (0,1)$

$$M(x;F) \begin{cases} = +\infty & \text{if } x \notin F, \\ < +\infty & \text{for almost all } x \in F. \end{cases}$$

4. Let $\{f_n\}$ be a sequence of measurable functions on X. Assume that

$$|f_n(x)| \leq g(x) \quad \text{for some integrable function } g. \tag{1}$$

(a) (5%) Show $\limsup \int f_n d\mu \leq \int_X \limsup f_n d\mu$.

(b) (10%) Is (a) still true if the condition (1) is dropped?

5. Let f be a real-valued function on $\mathbb{R}$.

(a) (10%) If f is Lebesgue measurable, then for any $a \in \mathbb{R}$, $f^{-1}(a)$ is a measurable set.

(b) (15%) Does the converse of (a) hold true? That is, if for any $a \in \mathbb{R}$, $f^{-1}(a)$ is measurable, then f is a measurable function.