104 學年度第二學期博士班資格考

Partial Differential Equations Exam **February, 2016**

This exam contains 9 problems with a total of 100 points. Show detailed argument to each problem.

1. (15 points) Let $a,\ b,\ c$ be three nonzero constants and let $f(s),\ g(s),\ h(s) : I \to \mathbb{R}$ be three differentiable functions defined on some open interval I containing $s_0 \in \mathbb{R}$. Consider the first-order linear equation

$$au_x(x,y) + bu_y(x,y) = c, \quad u = u(x,y).$$

   (a) (2 points) Find a sufficient condition so that the problem has a solution $u(x,y)$ defined on some open set $U \subset \mathbb{R}^2$ (the xy-plane) and satisfies $u(f(s), g(s)) = h(s)$ for all $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ for some $\varepsilon > 0$.

   (b) (3 points) Assume that $a = 1,\ b = 2,\ c = 3$. Find the general solution of the equation $u_x(x,y) + 2u_y(x,y) = 3$.

   (c) (6 points) Let $f(s) = s,\ g(s) = 2s,\ h(s) = 3s,\ s \in (1-\varepsilon, 1+\varepsilon)$. The curve $C(s) = (s, 2s, 3s),\ s \in (1-\varepsilon, 1+\varepsilon)$, is *everywhere* tangent to the characteristic direction $(1,2,3)$ (i.e. $C(s)$ is a *characteristic curve*). Find three different solutions $u(x,y)$ to the equation $u_x(x,y) + 2u_y(x,y) = 3$ with $u(s,2s) = 3s$ for all $s \in (1-\varepsilon, 1+\varepsilon)$.

   (d) (4 points) Let $f(s) = s,\ g(s) = 2s,\ h(s) = s^3,\ s \in (1-\varepsilon, 1+\varepsilon)$. The curve $C(s) = (s, 2s, s^3),\ s \in (1-\varepsilon, 1+\varepsilon)$, is tangent to the characteristic direction $(1,2,3)$ at $s_0 = 1$ (i.e. $C(s)$ is characteristic at $s_0 = 1$ and only at $s_0 = 1$). Is there a solution $u(x,y)$ to the equation $u_x(x,y) + 2u_y(x,y) = 3$ with $u(s,2s) = s^3$ for all $s \in (1-\varepsilon, 1+\varepsilon)$? Give your reasons.

2. (10 points) Consider the second order linear equation in two variables

$$u_{xx} - 4u_{xy} - 2u_{yy} = 0, \quad (x,y) \in \mathbb{R}^2, \quad u = u(x,y) : \mathbb{R}^2 \to \mathbb{R}.$$

   What is the type (elliptic, hyperbolic, or parabolic) of this equation? Find the general solution of the equation.

3. (10 points) Among the three equations

$$\Delta u(x) = 0, \quad u_t(x,t) = \Delta u(x,t), \quad u_{tt}(x,t) = \Delta u(x,t), \quad x \in \mathbb{R}^n, \quad t \in (-\infty, \infty)$$

   which one has the *smoothing effect* (i.e. a C^2 solution on $\mathbb{R}^n$, or on $\mathbb{R}^n \times (-\infty,\infty)$, is automatically a C^∞ solution)? **(just provide the answer for this question)**. If an equation has no smoothing effect, give a solution to that equation whose regularity is in C^2 only (but not in C^k for any $k \geq 3$).

4. (10 points) Again, for the three equations in the Problem 3, which one has the *Liouville property* (i.e. a C^2 solution on $\mathbb{R}^n$, or on $\mathbb{R}^n \times (-\infty,\infty)$, which is **bounded** above and below, must be a constant)? **(just provide the answer for this question)**. If an equation does not have the Liouville property, give an example to demonstrate this.

5. (10 points) Let $O \in \mathbb{R}^n$ be the origin of $\mathbb{R}^n$ and $B_{4R}(O)$ is the ball centered at O with radius $4R$, where $R > 0$ is a constant. Assume $u : B_{4R}(O) \to \mathbb{R}$ is a *nonnegative* harmonic function. Show that for any two points $p,\ q \in B_R(O)$ (ball with radius R) we have the estimate $u(p) \leq 3^n u(q)$ and derive the following **Harnack inequality** for *nonnegative* harmonic functions on $B_{4R}(O)$:

$$\sup_{B_R(O)} u \leq 3^n \inf_{B_R(O)} u.$$

Note that the constant 3^n is independent of the radius R and the function u. Hint: consider u on the balls $B_R(p)$, $B_{2R}(O)$, $B_{3R}(q)$.

6. (10 points) Let $\Omega \subset \mathbb{R}^{n3}$ be a bounded C^1 domain (open connected set). Assume that $u \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$, which is not a constant function and satisfies $u > 0$ on $\bar{\Omega}$ and $\frac{\partial u}{\partial n} = 0$ (Neumann condition) on $\partial\Omega$. Show that we have

$$\int_\Omega \left(\frac{\Delta u}{u}\right)(x)\, dx > 0.$$

7. (12 points) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C^2(\Omega) \bigcap C^0(\bar{\Omega})$. Consider the following six equalities and inequalities:

$$\begin{cases} (1).\ (\Delta u)(x) + u(x) = 0 \quad \text{in} \quad \Omega, \\ (2).\ (\Delta u)(x) - u(x) = 0 \quad \text{in} \quad \Omega, \\ (3).\ (\Delta u)(x) + u(x) \geq 0 \quad \text{in} \quad \Omega, \\ (4).\ (\Delta u)(x) - u(x) \geq 0 \quad \text{in} \quad \Omega, \\ (5).\ (\Delta u)(x) + u(x) \leq 0 \quad \text{in} \quad \Omega, \\ (6).\ (\Delta u)(x) - u(x) \leq 0 \quad \text{in} \quad \Omega. \end{cases}$$

Which one has the maximum principle (by this we mean that $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$)? Which one has the minimum principle (by this we mean that $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$)? Your answer should be, for example, like: (1) has the maximum principle and it also has the minimum principle. (2) does not have the maximum principle, but it has the minimum principle .... etc.

8. (10 points) Again, consider the Cauchy problem for the heat equation

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0 \quad \text{for} \quad x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = \phi(x) \quad \text{for} \quad x \in \mathbb{R}^n. \end{cases}$$

Now we assume that

$$\phi(x) \in C^0(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$$

for some $1 \leq p < \infty$. Show that there exists a constant $C > 0$, independent of (x,t), such that the solution $u(x,t)$ (the one given by the *convolution* of the fundamental solution and ϕ) satisfies the estimate

$$|u(x,t)| \leq \frac{C}{t^{\frac{n}{2p}}}$$

for all $x \in \mathbb{R}^n$, $t > 0$.

9. (13 points) Consider the Cauchy problem for the heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 \quad \text{for} \quad x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) \quad \text{for} \quad x \in \mathbb{R} \end{cases}$$

where $\phi(x)$ is the jump function given by

$$\phi(x) = 1, \quad \text{if } x \geq 0, \qquad \phi(x) = 0, \quad \text{if } x < 0.$$

(a) (8 points) Show that the solution $u(x,t)$ (the one given by the *convolution* of the fundamental solution and ϕ) satisfies

$$\lim_{t\to 0^+} u(0,t) = \frac{1+0}{2} = \frac{1}{2}.$$

(b) (5 points) Does the 2-dimensional limit

$$\lim_{(x,t)\to(0,0^+)} u(x,t)$$

exist or not? Give your reasons.