104 學年度第二學期博士班資格考

PhD Qualifying Exam in Numerical Analysis

Spring 2016

1. (15%) Let f be an $n+1$ times continuously differentiable real-valued function on $[a,b]$ and let $x_0, x_1, \cdots, x_n$ be $n+1$ distinct numbers in $[a,b]$.

   (a) Prove that there exists a unique polynomial P_n of degree at most n such that

   $$P_n(x_i) = f(x_i) \quad \text{for } i = 0, 1, \cdots, n.$$

   (b) Prove that for each x in $[a,b]$ there exists a $\xi_x \in (a,b)$ such that

   $$f(x) - P_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i).$$

2. (15%) Let f be a twice continuously differentiable real-valued function on $[a,b]$.

   (a) Let $x_0 = a$, $x_1 = b$ and $h = b - a$. Use the Lagrange interpolation to derive the trapezoid rule with an error term for $\int_a^b f(x)dx$.

   (b) Let $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of $[a,b]$ and let $h = (b-a)/n$ be the mesh size. Derive the composite trapezoid rule for approximating $\int_a^b f(x)dx$ and prove that the error term of the composite trapezoid rule is

   $$-\frac{1}{12}(b-a)h^2 f''(\xi), \quad \text{for some } \xi \in (a,b).$$

3. Consider the following two-point boundary value problem:

   $$-u''(x) + u(x) = f(x) \ \text{ for } \ 0 < x < 1, \quad u(0) = \alpha \text{ and } \ u(1) = \beta.$$

   Let $\mathcal{P}$ be a uniform mesh of the interval $[0,1]$ with the grid points $x_j = jh$, $0 \le j \le m+1$, where $h = 1/(m+1)$ is the mesh size.

   (a) (10%) Show that

   $$u''(x_i) = \frac{1}{h^2}\Big\{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})\Big\} - \frac{1}{12}h^2 u^{(4)}(\xi) \text{ for some } \xi \in (x_{i-1}, x_{i+1}).$$

   (b) (10%) Derive the centered difference scheme for solving the boundary value problem on the uniform mesh. Please write down explicitly the resulting linear system $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$, where $\boldsymbol{u} = [u_1, \cdots, u_m]^\top$ is the finite difference solution.

(c) (10%) Let $\widehat{\boldsymbol{u}} = [u(x_1), \cdots, u(x_m)]^\top$ be the vector of true values and $\boldsymbol{u} = [u_1, \cdots, u_m]^\top$ be the finite difference solution obtained in (b). Then the error vector E defined by $E = \widehat{\boldsymbol{u}} - \boldsymbol{u}$ contains the errors at each grid point. Prove the finite difference scheme derived in (b) convergent with the order of accuracy $O(h^2)$ in the discrete L^2-norm by showing that $\|\boldsymbol{A}^{-1}\|_2$ is uniformly bounded as $h \to 0^+$.

Hint: The eigenvalues of the $m \times m$ matrix

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{m\times m}$$

are given by $\lambda_p = -2(\cos(p\pi h) - 1)$ for $p = 1, 2, \cdots, m$.

(d) (10%) Let $\boldsymbol{Au} = \boldsymbol{f}$ be the linear system derived in (b). State the Jacobi iterative method in the matrix form for solving the linear system. Does the Jacobi iterative method converge for any initial guess $\boldsymbol{u}^{[0]} \in \mathbb{R}^m$? Give your reason.

4. Consider the following initial-boundary value problem for the 1-D heat equation:

$$\begin{aligned} &u_t = \nu u_{xx}, \quad 0 < x < 1,\ t > 0, \\ &u(0,t) = u(1,t) = 0, \quad t > 0, \\ &u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{aligned}$$

where constant $\nu > 0$ is the thermal conductivity. Let u_j^n be an approximation to $u(x_j, t_n)$, where $x_j = jh$, $t_n = nk$, k is the time step and h is the spatial grid size.

(a) (10%) Derive the Crank-Nicolson method for the 1-D heat problem and show that the local truncation error is second order accurate in both space and time, i.e., $O(h^2 + k^2)$.

(b) (10%) Show that the Crank-Nicolson method is unconditionally stable.

5. (10%) Consider the following Cauchy problem for the scalar advection equation:

$$\begin{aligned} &u_t + a u_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where $a > 0$ is a given constant. Let u_j^n be an approximation to $u(x_j, t_n)$, where $x_j = jh$, $t_n = nk$, k is the time step and h is the spatial grid size. Derive the Lax-Wendroff method, which is a second-order accurate scheme for numerically solving the Cauchy problem, by applying the Taylor expansion on $u(x, t+k)$ around (x, t). We can find that the scheme is a discretization of following advection-diffusion equation using the forward difference in time and centered difference in space,

$$u_t + a u_x = \frac{a^2 k}{2} u_{xx}.$$