DEPARTMENT OF MATHEMATICS
NATIONAL CHIAO-TUNG UNIVERSITY

Ph. D. Qualifying Examination

Spring, 2016.

ANALYSIS

Answer all of the following 8 questions. Each of Questions 1-6 carries 10 points, and each of Questions 7 and 8 carries 20 points.

1. Let f be a positive measurable function defined on $\mathbb{R}$. Show that there is a sequence $\{f_n\}_n$ of simple functions such that $f_n(x)$ monotonically increases to $f(x)$ everywhere.

2. Show that any nondecreasing real-valued function on $[0, 1]$ is Riemann integrable.

3. Prove that the derivative function f' of a real-valued differentiable function f on $[0, 1]$ is Lebesque measurable.

4. Let φ be a linear functional of the real Hilbert space $L^2[0, 2\pi]$ defined by

$$\varphi(f) = \int_0^{2\pi} f(2\pi - x)dx.$$

Prove that $||\varphi|| = \sqrt{2\pi}$, and find an g in $L^2[0, 2\pi]$ such that

$$\varphi(f) = \int_0^{2\pi} f(x)g(x)dx, \quad \forall f \in L^2[0, 2\pi].$$

5. Assume that every moment

$$m_n(f, [a, b]) = \int_a^b t^n f(t)\, dt, \quad \forall n = 0, 1, 2, \dots,$$

of a real-valued Legesgue integrable function f defined on $[a, b]$ is zero. Show that $f(t) = 0$ almost everywhere in $[a, b]$. What can you say for an integrable function f on $(-\infty, +\infty)$?

6. (Baire Category Theorem) Let X be a complete metric space and $\{X_n : n \in \mathbb{N}\}$ be a countable collection of closed subsets of X such that $X = \bigcup_n X_n$. Prove that at least one of X_n's has non-empty interior.

7. An extended real valued function $f : \mathbb{R} \longrightarrow [-\infty, +\infty]$ is said to be *lower semi-continuous* at the point y if

$$f(y) \neq -\infty \quad \text{and} \quad f(y) \leq \liminf_{x \to y} f(x).$$

Show the following statements.

(a) Let $f(y)$ be finite. Then f is lower semicontinuous at y if and only if given $\epsilon > 0$, there is a $\delta > 0$ such that $f(y) \leq f(x) + \epsilon$ for all x with $|x - y| < \delta$.

(b) A real valued function f is lower semicontinuous on (a, b) if and only if the set $\{x \in \mathbb{R} : f(x) > \lambda\}$ is open for each real number λ.

(c) A lower semicontinuous real valued function f defined on $[a, b]$ bounded from below assumes its minimum on $[a, b]$.

(d) (Dini Theorem) Let $\{f_n\}_n$ be a sequence of lower semicontinuous functions defined on $[a, b]$. Suppose $f_n(x)$ monotonically increasing to 0 for all x in $[a, b]$. Then f_n converges to zero uniformly on $[a, b]$.

8. Let $I = [0, 1]$ and $Q = I \times I$. Define $f : Q \longrightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{1}{p} & \text{if } y \text{ is rational and } x = \frac{q}{p}, p, q \in \mathbb{N} \text{ such that} \\ & \text{the greatest common factor } (p, q) \text{ of } p \text{ and } q \text{ is } 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is f Riemann integrable over Q? If yes, compute $\int_Q f$.

(b) For each fixed x, compute the lower Riemann integral $\underline{\int}_{y \in I} f(x, y)$ and the upper Riemann integral $\overline{\int}_{y \in I} f(x, y)$.

(c) Show that $\int_{y \in I} f(x, y)$ exists for x in $I - D$, where D is a set of measure zero in I.

(d) Verify Fubini's theorem for $\int_Q f$.