104 學年度第二學期博士班資格考

Algebra Qualifying Exam

- Let $\mathbb{Z}$ be the ring of integers.
- Let $\mathbb{Q}$ be the field of rational numbers.
- Let $\mathbb{C}$ be the field of complex numbers.

1. (20 points) Let G be a group of order 168. Suppose that G is simple.
   (a) Let P be a Sylow 7-subgroup of G. Let $\mathrm{N}_G(P) = \{g \in G \mid g^{-1}Pg = P\}$ be the normalizer of P in G. Find the order of $\mathrm{N}_G(P)$.
   (b) Show that G has no element of order 14.
   (c) If H is a proper subgroup of G, show that $|G : H| \geq 7$.

2. (20 points) Prove or disprove each of the following statements.
   (a) $1 + \sqrt{-23}$ is a prime element in $\mathbb{Z}[\sqrt{-23}] = \{a + b\sqrt{-23} \mid a, b \in \mathbb{Z}\}$.
   (b) $\mathbb{Z}[\sqrt{-23}]$ is a unique factorization domain (UFD).
   (c) If S is a principal ideal domain (PID), then the polynomial ring $S[x]$ is a PID.

3. (20 points) Let R be a commutative ring with identity such that every submodule of every free R-module is free.
   (a) Is it true that every ideal of R is a principal ideal?
   (b) Is it true that R is an integral domain?

4. (20 points)
   (a) Construct a finite field F of order 27.
   (b) Let $S = \{r^2 \mid r \in F\}$ be the set of squares in F. Find the number of elements in S.
   (c) Show that every element in F can be written as a sum of two squares.

5. (20 points) Let $f(x) = x^4 - 7$ be a polynomial over $\mathbb{Q}$. Suppose that $E \subseteq \mathbb{C}$ and E is the splitting field of $f(x)$ over $\mathbb{Q}$.
   (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
   (b) Let $i = \sqrt{-1}$ and let $\alpha = \sqrt[4]{7}$ be the unique positive real root of $x^4 - 7$. Show that $E = \mathbb{Q}(\alpha, i)$.
   (c) Determine $[E : \mathbb{Q}]$.
   (d) Let $K = \mathbb{Q}(\sqrt{7})$. Determine the Galois group $\mathrm{Gal}(E/K)$.