NCTU Department of Applied Mathematics
Discrete Mathematics Qualifying Examination, September 2015

NOTE: ***Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.***

Problem 1.(15pts) (a) State P. Hall's marriage theorem. (b) Prove P. Hall's marriage theorem from the max-flow min-cut theorem (this theorem is also referred to as the the Ford-Fulkerson theorem).

Problem 2.(15pts) Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \text{ I})$ be an incidence structure with $|\mathcal{P}| = |\mathcal{B}| = v$, block-size k, such that any two blocks meet in λ points. Then $\mathcal{D}$ is a symmetric 2-design.

Problem 3.(10pts) State and prove the Erdös-Ko-Rado theorem.

Problem 4.(15pts) The Möbius function $\mu : \mathbb{N} \to \mathbb{C}$ is defined by

$$\mu(n) = \begin{cases} 0 & \text{if } p^2 | n \text{ for some prime } p, \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

For a function $f : \mathbb{N} \to \mathbb{C}$, the function $\widehat{f} : \mathbb{N} \to \mathbb{C}$ is defined by $\widehat{f}(n) = \sum_{d|n} f(d)$.

(a) Prove that

$$\widehat{\mu}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) State and prove the Möbius inversion formula by using the above notation.

(c) The Riemann Zeta Function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is defined in the complex plane for $\text{Re}(s) > 1$. Prove that $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$.

Problem 5.(15pts) Consider a nonempty set with a non-associative (binary) product operation. If $a_1, a_2, \ldots, a_n$ are all distinct elements in this set, then we must *parenthesize* the expression $a_1 a_2 \cdots a_n$ in order to indicate the product. Prove that the number of ways to parenthesize a product of n terms is the n^{th} Catalan Number $C_n = \frac{1}{n}\binom{2n-2}{n-1}$.

Problem 6.(15pts) Let G be a planar graph with n vertices $v_1, v_2, \ldots, v_n$. For $i = 1, 2, \ldots, n$, let L_i be a set of 5 distinct colors. Prove that there exists a mapping f on the vertices of G with $f(v_i) \in L_i$ such that $f(v_i) \neq f(v_j)$ for all adjacent pairs v_i, v_j.

Problem 7.(15pts) A *tournament* T_n is an orientation of K_n. A tournament is *transitive* if, whenever $x \to y$ and $y \to z$, then $x \to z$ also. Note that every tournament T_n $(n \geq 4)$ contains at least one transitive subtournament T_3, but not every tournament T_n is itself transitive.

(a) Prove that if $k \leq \log_2 n$, then every tournament T_n has a transitive subtournament T_k.

(b) Prove that if $k > 1 + 2\log_2 n$, then there exists a tournament on n vertices with no transitive subtournament on k vertices.