Qualifying Exam.: Ordinary Differential Equations **September 16, 2015**

This exam. contains 5 problems with a total of 100 points.
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1. Let $\varphi_t(x)$ be the solution of the IVP $dx/dt = f(x) \in C^1$, $\varphi_0(x_0) = x_0$. Assume that the point x_0 generates a periodic orbit under $dx/dt = f(x)$ with period $T_1 > 0$ ($\varphi_{T_1}(x_0) = x_0$) and with period $T_2 > 0$ ($\varphi_{T_2}(x_0) = x_0$). Show that:

(4 points) (a) x_0 generates a periodic orbit with period $mT_1 + nT_2$ for any integers m, n.

(6 points) (b) If x_0 is not an equilibrium point, then there is a minimal period $T_0 > 0$ where $T_1/T_0, T_2/T_0$ are both integers.

(5 points) (c) If T_1/T_2 is irrational, then x_0 must be an equilibrium.

(5 points) (d) If $T_1/T_2 = a/b$ is rational ($a, b \neq 0$ and a, b are positive integers with no common factor), then x_0 generates a periodic orbit with period T_2/b.

2. Consider the RLC circuit equation:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0,$$

where constants

$$C, L > 0 \text{ and } R \geq 0.$$

(2 points) (a) Rewrite the RLC circuit equation as a two-dimensional linear system on the (I, J)-plane, where $J := \dfrac{dI}{dt}$.

(8 points) (b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.

(10 points) (c) Classify the fixed point at the origin and draw possible phase portraits on the (I, J)-plane.

3. Consider

$$\begin{cases} x' = -y - \beta x + (3x^2 + 2y^2)x \\ y' = x - \beta y + (3x^2 + 2y^2)y, \quad \beta \in \mathbf{R}. \end{cases} \tag{1}$$

(10 points) (a) Show that (1) has at least one periodic orbit if $\beta > 0$.

(5 points) (b) Discuss the existence of periodic orbits of (1) for $\beta \leq 0$. Justify your answer.

4. (15 points) Consider the nonautonomous linear system

$$x' = A(t)x,$$

where $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ is a continuous, T-periodic matrix (i.e., $A(t) = A(t+T)$ for all t). Let $\Phi(t)$ be a fundamental matrix of matrix equation

$$X' = A(t)X \tag{2}$$

where $X = (x_{ij}(t)) \in \mathbb{R}^{n\times n}$ and $X' = \left(x'_{ij}(t)\right)$. **Prove Floquet's Theorem.**

(Floquet's Theorem) If $\Phi(t)$ is a fundamental matrix of (2), then so is $\Phi(t+T)$. Moreover there exists $P(t) \in \mathbb{C}^{n\times n}$ which is nonsingular and satisfies $P(t) = P(t+T)$ for all t, and there exists $R \in \mathbb{C}^{n\times n}$ such that

$$\Phi(t) = P(t)e^{tR}.$$

(Please turn this page over and continue with Problems 5, Page 2.)

5. (5 points) (a) State LaSalle's invariance principle for the IVP

$$x' = f(x), \quad x(0) = x_0.$$

(b) Consider the Lotka-Volterra two species competition model

$$\begin{cases} x_1' &= r_1 x_1 \left(1 - \dfrac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \\ x_2' &= r_2 x_2 \left(1 - \dfrac{x_2}{K_2}\right) - \alpha_2 x_1 x_2 \end{cases} \tag{3}$$

with constants $r_1, r_2, K_1, K_2, \alpha_1, \alpha_2 > 0$. Assume that

$$\frac{r_1}{\alpha_1} > K_2 \text{ and } \frac{r_2}{\alpha_2} > K_1 \quad \text{(the weak competition case)}.$$

(10 points) (i) Show that nonlinear system (3) has a unique positive equilibrium point (x_1^*, x_2^*) and it is a stable node for nonlinear system (3).

(10 points) (ii) Apply LaSalle's invariance principle to show that the positive equilibrium (x_1^*, x_2^*) is globally asymptotically stable.

(5 points) (iii) Draw the phase portrait of nonlinear system (3).