Department of Mathematics
National Chiao-Tung University

Ph. D. Qualifying Examination

Fall, 2015.

Analysis

Answer all of the following questions. Each question carries equally 10 points.

1. Show that any nondecreasing real-valued function on $\mathbb{R}$ can have at most countably many discontinuous points.

2. Assume a real-valued integrable function f defined on $[a, b]$ satisfies that
$$\int_a^x f(t)\,dt = 0$$
for all x in $[a, b]$. Show that $f(t) = 0$ almost everywhere in $[a, b]$.

3. Prove that if a subset A of $[0, 1]$ has measure zero then
$$A^2 = \{x^2 : x \in A\}$$
has measure zero, too. If f is a continuously differentiable function on $[0, 1]$, can you conclude again that $f(A)$ has measure zero? Justify your answer.

4. We call a series $\sum_{n=1}^{\infty} a_n$ of real numbers *unconditionally summable* if the sum $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges for every permutation of positive integers, i.e., a one-to-one and surjective transformation $\sigma : \mathbb{N} \to \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} a_n$ is unconditionally summable if and only if it is absolutely summable, i.e., $\sum_{n=1}^{\infty} |a_n| < +\infty$. In this case, show that
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}, \quad \text{for every permutation } \sigma \text{ of } \mathbb{N}.$$

5. Let P be the class of all even polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^{2j}.$$

Show that a continuous function f in $C[-\pi, \pi]$ can be uniformly approximated on $[-\pi, \pi]$ by a sequence in P if and only if $f(-x) = f(x)$.

6. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous and of bounded variation. Assume that f is absolutely continuous on $[\epsilon, 1]$ for each $\epsilon > 0$. Show that f is absolutely continuous on $[0,1]$.

7. Let $f_n \to f$ on $[0,1]$ in the following sense: for every x in $[0,1]$, if $x_n \to x$, then $f_n(x_n) \to f(x)$. Show that f is continuous if all f_n are continuous.

8. Let $\{G_n\}_n$ be a sequence of non-empty open sets in $[0,1]$ with the Lebesgue measures $m(G_n) \leq 1/2^n$ for $n = 1, 2, \ldots$. Let

$$f(x) = \sum_{n=1}^{\infty} m(G_n \cap [0,x]), \quad 0 \leq x \leq 1.$$

Show that f is continuous, non-decreasing, and that $f'(x) = +\infty$ for all x in $\cap_{n=1}^{\infty} G_n$.

9. Prove or disprove: There exist continuous real-valued functions f and g defined on $[0,1]$ such that $f(x) = g(x)$ for uncountably many points x, but in every interval there exists a point x where $f(x) \neq g(x)$.

10. Let $\{f_n\}_n$ be a sequence of functions in the separable infinite dimensional Hilbert space $L^2[0, 2\pi]$. Suppose that

$$\lim_{n\to\infty} \int_0^{2\pi} f_n(x) \cos mx \, dx = \lim_{n\to\infty} \int_0^{2\pi} f_n(x) \sin mx \, dx = 0, \quad \forall m = 0, 1, 2, \ldots.$$

Show that there is a constant $M > 0$ such that

$$\int_0^{2\pi} |f_n(x)|^2 \, dx \leq M, \quad \forall n = 1, 2, \ldots.$$

If $\lim_{n\to\infty} f_n(x) = f(x)$ pointwisely on $[0, \pi]$, what is f?