

國立交通大學應用數學系
博士班離散數學資格考
Spring 2015

Problem 1. Let b be the number of blocks of an $S_\lambda(t, k, v)$ with $t \geq 2s$ and $v \geq k + s$.

(1) (5pts) Prove that $b = \frac{\lambda \binom{v}{t}}{\binom{k}{t}}$. (2) (10pts) Prove that $b \geq \binom{v}{s}$.

Problem 2. (15pts) Prove that there are exactly five Platonic solids by using Euler's formula.

注釋：正多面體 (Platonic solid) 是指每一個頂點所接的面數都是一樣的凸多面體，且其各面都是全等的正多邊形。

Problem 3. (15pts) Suppose that the edges of K_n are colored red or blue, and let Δ denote the number of monochromatic triangles. Prove that $\Delta \geq \binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor \left(\frac{n-1}{2} \right)^2 \right\rfloor \right\rfloor$.

Problem 4. (15pts) Let the points $1, 2, \dots, 2n$ be on a circle (consecutively). Find the number of ways of joining them in pairs by n nonintersecting chords. Include all the details of the proof.

Problem 5. (15pts) State and prove the Burnside's lemma.

Problem 6. (15pts) Let the sets $A_i, 1 \leq i \leq k$, be distinct subsets of $\{1, 2, \dots, n\}$. Suppose $A_i \cap A_j \neq \emptyset$ for all i and j . Prove that $k \leq 2^{n-1}$.

Problem 7. (10pts) Suppose G is a graph on n vertices with more than $\frac{n\sqrt{n-1}}{2}$ edges. Prove that G has girth ≤ 4 .