Qualifying Exam.: Ordinary Differential Equations **March 4, 2015**

1.(15%) Let $x(t)$ be a positive and continuous function defined in $[a, b]$ such that

$$x(t) \leq M + \int_a^t g(s) f(x(s)) ds, \quad \text{for } t \in [a, b],$$

where $M \geq 0$, $g(t) : [a, b] \to \mathbb{R}^+$ is continuous and $f(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and monotonic increasing. Prove that

$$x(t) \leq h^{-1}\big(h(M) + \int_a^t g(s) ds\big), \quad \text{for } t \in [a, b],$$

where $h(u) : \mathbb{R} \to \mathbb{R}$ is defined by $h(u) = \int_{u_0}^{u} \frac{1}{f(s)} ds$ for some constant $u_0 \in \mathbb{R}$.

2. (20%) (1) (10%) Consider the equation:

$$(2 - t)y''' + (2t - 3)y'' - ty' + y = 0, \quad t < 2.$$

Find the general solution of the equation by using the particular solution e^t.

(2) (10%) Find the general solution of the following non-homogeneous system:

$$\begin{cases} x' = 7x - y + 6z - 5t - 6, \\ y' = -10x + 4y - 12z - 4t + 23, \\ z' = -2x + y - z + 2. \end{cases}$$

3. (30%) (1) (10%) Consider the system:

$$\begin{cases} x' = \mu x + y - x(x^2 + y^2), \\ y' = -x + \mu y - y(x^2 + y^2), \end{cases}$$

where $\mu > 0$ is a constant. Show that the system has a unique equilibrium, and a stable periodic solution.

(2) (20%) Consider the system:

$$\begin{cases} x' = \gamma_1 x(1 - \frac{x}{K_1}) - \alpha_1 xy, \\ y' = \gamma_2 y(1 - \frac{y}{K_2}) - \alpha_2 xy, \\ \gamma_1, \gamma_2, K_1, K_2, \alpha_1, \alpha_2, x(0), y(0) > 0. \end{cases}$$

Determine the local stability of all possible non-negative equilibrium, and prove that there is no periodic solution in the first quadrant.

4. (25%) (1) (15%) Consider the system:

$$\begin{cases} x' = x - y - x^3 - xy^2, \\ y' = x + y - x^2 y - y^3, \\ z' = \lambda z, \end{cases}$$

where λ is a real constant.

(i) (5%) Show that the system has a periodic solution $\gamma(t) := (\cos t, \sin t, 0)$.

(ii) (10%) Compute the Poincaré map for $\gamma(t)$, and the Floquet's (characteristic) exponent of $\gamma(t)$.

(2) (10%) Let $a(t)$ and $b(t)$ be continuous and T-periodic functions, $\phi_1(t)$ and $\phi_2(\mathrm{t})$ be solutions of $y'' + a(t)y' + b(t)y = 0$ such that

$$\phi_1(0) = 1,\ \phi_2(0) = 0,\ \phi_1'(0) = 0,\ \phi_2'(0) = 1.$$

Show that the Floquet's multipliers λ satisfy $\lambda^2 + \alpha\lambda + \beta = 0$, where

$$\alpha := -[\phi_1(T) + \phi_2'(T)] \text{ and } \beta := \exp(-\int_0^T a(t)dt).$$

5. (10%) Consider the system:

$$\begin{cases} x' = x(1 - x - y), \\ y' = y(\frac{3}{4} - y - \frac{1}{2}x). \end{cases}$$

Use a suitable Liapunov function to show that the equilibrium (1/2,1/2) is asymptotically stable.