博士班資格考

103年9月

Probability, September 2014

Let $\{X_n\}$ denote a sequence of random variables and $S_n = \sum_{k=1}^{n} X_k$ throughout.

1) Construct a strictly positive martingale M_k with $M_k \to 0$ a.s. as $k \to \infty$, and verify your answer. (15 points)

2) $\{X_n\}$ are *i.i.d.* with mean 0 and variance 1.
Find the limits and verify your answers. (10 points and 20 points respectively)

(2.1) $E\frac{|S_n|}{\sqrt{n}} \to$?, as $n \to \infty$.
(2.2) For a symmetric random walk, i.e. $P\{X_1 = 1\} = P\{X_1 = -1\} = \frac{1}{2}$,

$$\frac{\sum_{k=1}^{n} f(S_k)}{n} \to \ ? \ \text{in } L^1, \ \text{as } n \to \infty,$$

where f is a real-valued function defined on integers with $\sum_{k=-\infty}^{\infty} | f(k) | < \infty$.
(Hint: $n! \sim \sqrt{2\pi} \ n^{n+\frac{1}{2}} \ e^{-n}$.)

3) Let $\{A_n\}$ be a sequence of pairwise independent events. If $\sum_{n=1}^{\infty} P\{A_n\} = \infty$, then $P\{A_n \ \ i.o.\} = 1$. (25 points)
(Hint: Let $Z_n = \sum_{k=1}^{n} I_{A_k}$. Prove that a subsequence of $\frac{Z_n}{EZ_n}$ converges to 1 *a.s.*)

4) $\{X_n\}$ are pairwise independent and identically distributed.

Prove that if $\dfrac{S_n - cn}{n^{1/p}}$ converges *a.s.* as $n \to \infty$, for some $c \in R$ and $p > 0$,

then $E \mid X_1 \mid^p < \infty$. (20 points)

5) Let $X(t)$, $t \geq 0$, be such that for any bounded stopping time τ, $X(\tau)$ is integrable and $\mathrm{E}X(\tau) = \mathrm{E}X(0)$. Prove that $X(t)$ is a martingale. (10 points)