博士班資格考

103.9月

DEPARTMENT OF APPLIED MATHEMATICS
CHIAO TUNG UNIVERSITY
Ph. D. Qualifying Examination
Sep, 2014
Analysis
(TOTAL 100 PTS, two pages)

Throughout this exam, dx and $|\cdot|$ represent the Lebesgue measure on $\mathbb{R}^n$, and χ_E is the characteristic function of the set E.

1. (50%) Prove or disprove the following statements:

(a) Let $f \in L^1(\mathbb{R})$ and $h > 0$. Then

$$\int_{-\infty}^{\infty} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt \right| dx \leq \int_{-\infty}^{\infty} |f(x)| dx.$$

(b) Let $f : [a,b] \mapsto \mathbb{R}$ be of bounded variation. Then $f' \in L^1[a,b]$ and

$$\int_a^b f'(x)dx = f(b) - f(a).$$

(c) Let f_k and f be Lebesgue measurable functions defined on $\mathbb{R}^n$ such that for any compact subset $\Omega \subset \mathbb{R}^n$,

$$\int_\Omega |f_k - f| dx \longrightarrow 0 \qquad \text{as } k \to \infty.$$

Then $f_k \longrightarrow f$ in $L^1(\mathbb{R}^n)$.

(d) Let $T : L^2[-\pi,\pi] \mapsto L^2[-\pi,\pi]$ be a bounded linear operator with the property: $T(\chi_E) = \alpha(\chi_E)^2$ for all Lebesgue measurable subsets E of $[-\pi,\pi]$, where α is a fixed constant. Then $T(f) = \alpha(f)^2$ for all $f \in L^2[-\pi,\pi]$.

(e) Let $0 < p \leq q < \infty$. Then $\ell^p \subset \ell^q$ and $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$.

2. (10%) Find the value:

$$\sup_{\|f\|_2 \neq 0} \frac{\left| \int_0^1 \frac{f(x)}{1+x^2} dx \right|}{\left(\int_0^1 |f(x)|^2 dx \right)^{1/2}}.$$

3. (10%) Let $0 < p \leq q < \infty$ and $a_n \geq 0 \quad (n = 1, 2, \cdots)$. Prove that

$$\left(\liminf_{n\to\infty} \frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \right)^{1/p} \leq \left(\limsup_{n\to\infty} \frac{a_1^q + a_2^q + \cdots + a_n^q}{n} \right)^{1/q}.$$

4. (10%) Let $\mathcal{P}$ be the set of polynomials. Prove that

$$\sup_{f\in C[0,1]}\left(\max_{x\in[0,1]}\frac{|f(x)|}{1+|f(x)|}\right)=\sup_{f\in\mathcal{P}}\left(\max_{x\in[0,1]}\frac{|f(x)|}{1+|f(x)|}\right).$$

5. (10%) Let $f, f_n \in L^1(\mathbb{R})$ for $n = 1, 2, \cdots$. Suppose that

$$\int_{-\infty}^{\infty}|f_n(x)-f(x)|^5dx \le \frac{1}{n^{3/2}} \qquad (n\ge 1).$$

Does $f_n(x)$ converge to $f(x)$ a.e.? Verify your result.

6. (10%) Let $A = \{x \in [0,1] : x = 0.a_1a_2\cdots \text{ and } a_n = 3 \text{ or } 4 \text{ for all } n\}$. Is A Lebesgue measurable ? If so, compute $|A|$.