Probability Qualifying Examination
September 2013

There are 10 problems. You turn in solutions for *exactly* 6 (your best 6). Indicate which 6 you have chosen on the front page of your answer book. Each problem is worth 20 points.

1. A is a Borel measurable subset of $[0,1]^2$. For each $x \in R$, define
$$A_x = \{y \in [0,1]; (x,y) \in A\}.$$
   (a) Prove A_x is a Borel subset of R for each $x \in R$.
   (b) Define
$$f(x) = \text{ Lebesgue measure of } A_x.$$
   Prove f is Borel measurable function on $[0,1]$.

2. Assume $X_1, X_2, \cdots$ are iid random variables such that $E[|X_1|] = \infty$. Prove
$$P(|X_n| \geq n \; i.o.) = 1.$$
   Here *i.o.* stands for infinitely often.

3. Let $X_1, X_2, \cdots$ be iid random variables such that
$$nP(|X_1| > n) \to 0.$$
   Define
$$S_n = X_1 + X_2 + \cdots + X_n$$
   and
$$\mu_n = E[X_1, |X_1| \leq n].$$
   Prove
$$\frac{S_n}{n} - \mu_n \to 0 \text{ in probability}$$
   as $n \to \infty$ (Note that we do not assume the finiteness of $E[|X_1|]$). In particular, if $E[|X_1|] < \infty$, then
$$\frac{S_n}{n} \to \mu \text{ in probability}$$
   as $n \to \infty$, where $\mu = E[X_1]$.

4. (a) Prove
$$(\frac{1}{x} - \frac{1}{x^3}) \exp(-\frac{x^2}{2}) \leq \int_x^\infty e^{-\frac{y^2}{2}} dy \leq \frac{1}{x} \exp(-\frac{x^2}{2}).$$
   (b) Let $X_1, X_2, \cdots$ be independent identically distributed standard normal random variables. Prove there is constant c such that
$$\limsup_n \frac{|X_n|}{\sqrt{\log n}} = c, \text{ almost surely.}$$

5. Assume $X_1, X_2, \cdots$ are independent Poisson random variables with $E[X_n] = \lambda_n$. Define

$$a_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$
$$S_n = X_1 + X_2 + \cdots + X_n.$$

Assume λ_n is bounded and

$$\sum_n \lambda_n = \infty.$$

Prove

$$\lim_{n\to\infty} \frac{S_n}{a_n} = 1 \text{ almost surely .}$$

(**Hint:** Denote $n_k = \inf\{n; a_n \geq k^2\}$ and consider $\frac{S_{n_k}}{a_{n_k}}$ and also $\frac{S_n}{a_n}$, $n_k \leq n < n_{k+1}$ separately.)

6. Assume $X_1, X_2, \cdots$ are independent random variables such that $E[X_n] = 0$ for all n and

$$\sum_n var(X_n) < \infty.$$

(a) Prove $\sum_n X_n$ converges in probability.

(b) Define

$$S_n = \sum_{i=1}^n X_i.$$

Prove $S_n, n = 1, 2, \cdots$ is a martingale. Show that the martingale convergence theorem is applicable to S_n and show S_n converges almost surely (a stronger result of (a)).

7. (a) Let ϕ be the characteristic function of a probability measure μ on R. Prove the inequality

$$\mu(|x| > 2/u) \leq \frac{1}{u}\int_{-u}^{u}(1 - \phi(t))dt, \ u > 0.$$

(b) Let Y_n be random variables with characteristic functions ϕ_n. Prove Y_n converges to 0 in probability as $n \to \infty$ if and only if $\phi_n(t) \to 1$, $n \to \infty$ for all $|t| < \delta$ for some $\delta > 0$.

8. Let $\{Y_n, n \geq 0\}$ be a Markov chain on $Z = \{1, 2, \cdots\}$ with transition probability $(p_{ij}, i, j \in Z)$. For a bounded function f on Z and $i \in Z$, define

$$Pf(i) = \sum_{j\in Z} p_{ij} f(j).$$

(a) Prove

$$S_n = f(Y_n) - \sum_{i=0}^{n-1} h(Y_i)$$

is a martingale with respect to $\{\mathcal{F}_n,\ n \geq 0\}$, where

$$\mathcal{F}_n = \sigma(Y_0, Y_1, \cdots, Y_n)$$

the σ field generated by $Y_0, Y_1, \cdots, Y_n$ and $h = Pf - f$.

(b) Prove

$$Z_n = S_n^2 - \sum_{i=0}^{n-1} g(Y_i)$$

is a martingale, where

$$g(i) = Pf^2(i) - (Pf(i))^2.$$

9. Let $X_1, X_2, \cdots$ be independent random variables. Define

$$\mathcal{G}_n = \sigma(X_n, X_{n+1}, \cdots)$$

the σ field generated by $X_n, X_{n+1}, \cdots$. The tail σ field is given by

$$\mathcal{T} = \cap_n \mathcal{G}_n.$$

(a) Prove for any $A \in \mathcal{T}$, $P(A) = 0$ or $P(A) = 1$.

(b) Prove $\sup_n X_n < \infty$ almost surely if and only if $\sum_n P(X_n > A) < \infty$ for some A.

10. Let f be a continuous function defined on $[0, 1]$. Define

$$f_n(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k f(\frac{k}{n}).$$

Prove

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \to 0,\ n \to \infty.$$