2013 Fall, ODE Qualifying Exam, National Chiao Tung University
Each problem is weighted of 20 points with total 100 points (5 problems).

1. Let $y(t), a(t), b(t)$ be three continuous nonnegative functions defined on $[0,1]$. Suppose that

$$y(t) \leq a(t) + \int_0^t b(s) y^p(s) ds, \quad t \in [0,1],$$

for some $p \in (0,1)$. Set $q = 1-p$. Show that

$$y(t) \leq a(t) + k_0^p \left(\int_0^t b^{1/q}(s) ds \right)^q, \quad t \in [0,1],$$

where $x = k_0$ is the unique positive root of the equation

$$x = \left[\int_0^1 a(s) ds \right] + \left\{ \int_0^1 \left[\int_0^t b^{1/q}(s) ds \right]^q dt \right\} x^p.$$

2. Consider the system

$$\begin{cases} x' = x(1-x-ky) \\ y' = ry(1-hx-y) \end{cases}$$

where r, h, k are positive constants. Determine the stabilities of the equilibria in the square $\{0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ for the following 3 cases:

$$h, k > 1; \quad h, k < 1; \quad 0 < h < 1 < k,$$

for any given $r > 0$. Justify your answers.

3. Let a_1, a_2 be continuous periodic functions of period ω. Let φ_1 and φ_2 be solutions of

$$x'' + a_1(t) x' + a_2(t) x = 0 \tag{1}$$

such that

$$\varphi_1(0) = 1,\ \varphi_1'(0) = 0,\ \varphi_2(0) = 0,\ \varphi_2'(0) = 1.$$

Show that the Floquet multipliers of the associated system to (1) are solutions of

$$\lambda^2 - A\lambda + B = 0,$$

where

$$A = \varphi_1(\omega) + \varphi_2'(\omega), \quad B = \exp\left[-\int_0^\omega a_1(t)\, dt\right].$$

4. Prove that the function

$$V(x,y) = y^{2m} + Ax^{2n} + A_1 x^{2n-1} y + \cdots + A_{2n-1} x y^{2n-1} + A_{2n} y^{2n},$$

where $A_1, \cdots, A_{2n}$ are real numbers and m, n are positive integers, is **positive definite** in a neighborhood of $(0,0)$ if $A > 0$ and $m < n$.

5. Consider the system

$$\begin{cases} x' = y \\ y' = -Ax^3 - Bx^2y - Cxy^2 - Dy^3 \end{cases}$$

where A, B, C, D are real constants.
(a) Show that $(0,0)$ is unstable if $A < 0$. (Hint: consider $V(x,y) = xy$)
(b) Show that $(0,0)$ is asymptotically stable if $A > 0$ and $B > 0$.
(Hint: consider $V(x,y) = \frac{y^2}{2} + \frac{A}{4}x^4 + b_1x^3y + b_2x^2y^2 + b_3xy^3 + b_4y^4$)
State also the theorems you used in each case.