DEPARTMENT OF APPLIED MATHEMATICS
CHIAO TUNG UNIVERSITY
Ph. D. Qualifying Examination
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Analysis
(TOTAL 100 PTS, two pages)

Throughout this exam, $\|x\|$ denotes the norm of x, $B(0;r) = \{x \in \mathbb{R}^n : \|x\| < r\}$, dx and $|\cdot|$ represent the Lebesgue measure on $\mathbb{R}^n$, and χ_E is the characteristic function of the set E.

1. (50%) Prove or disprove the following statements:

   (a) Any finite Borel measure μ defined on $\mathbb{R}^n$ is of the form: $\mu = f(x)dx + \nu$ for some $f \in L^1(\mathbb{R}^n)$ and $\nu \perp dx$.

   (b) Let E be a Lebesgue meaurable subset of $\mathbb{R}^3$. Then

   $$|E| = \sup_{r>0} |E \cap B(o,r)|.$$

   (c) Let $f_n, f \in L^2[-\pi,\pi]$ for $n \geq 1$, and all of f_n, f be real-valued. If $\|f_n\|_2 \to \|f\|_2$ and $f_n \to f$ weakly, then $f_n \to f$ in $L^2[-\pi,\pi]$.

   (d) Let $f \in C[0,1]$ and $\epsilon > 0$. Then there exists a polynomial g such that $\sup_{0\leq x\leq 1} |f(x^2) - g(x^2)| < \epsilon$.

   (e) Let $0 < p < q < \infty$. Then $\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p} \geq \left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{1/q}$.

2. (10%) Let μ, ν be two finite Borel measures defined on $[0,1]$ with

   the property:

   $$\int_E e^{-x} d\mu = \int_E x e^{-x} d\nu \qquad \text{for all Borel sets } E \subset [0,1].$$

   Can you conclude $d\mu = x d\nu$? that is,

   $$\int_0^1 f(x) d\mu = \int_0^1 x f(x) d\nu \qquad (f \geq 0).$$

   Give your reason.

3. (10%) Let $\alpha > -1$ and $\omega_\alpha(x) = x^\alpha (1 + e^x)^{-1}$. Prove that

   $$\int_0^\infty \omega_\alpha(x) dx = \sum_{n=1}^\infty (-1)^{n+1} \int_0^\infty e^{-nx} x^\alpha dx.$$

4. (10%) Let $y_m = \sum_{n=1}^{\infty} a_{mn} x_n$, where $a_{mn} \in \mathbb{C}$ for $m, n \geq 1$.

(a) Prove that $T : x \mapsto y$ defines an operator from ℓ^2 to ℓ^1, where $x = \{x_n\}_{n=1}^{\infty} \in \ell^2$ and $y = \{y_m\}_{m=1}^{\infty} \in \ell^1$.

(b) Prove that $\|T\| \leq \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} |a_{mn}|^2 \right)^{1/2}$.

5. (10%) Let $g \geq 0$ be a Lebesgue measurable function defined on $[0, 1]$.

Show that

$$\int_0^1 \int_{x^2}^1 g(t)dtdx = \int_0^1 \sqrt{t} g(t)dt.$$

6. (10%) Let $p \geq 1$, $0 < a < b < \infty$ and $F(x) = \left(\int_x^b f(t)dt \right)^p$, where

$f \in L^1[a, b]$.

(a) Prove that $F(x)$ is absolutely continuous on $[a, b]$.

(b) Deduce the following equality:

$$\left(\int_a^b f(t)dt \right)^p = p \int_a^b \left(\int_x^b f(t)dt \right)^{p-1} f(x)dx.$$