Qualifying Exam.: Ordinary Differential Equations **February 21, 2013**

This exam. contains 5 problems with a total of 100 points.

1. Consider the Lotka-Volterra two-species competition model

$$\begin{cases} \dfrac{dx}{dt} = r_1 x\left(1 - \dfrac{x}{K_1}\right) - \alpha_1 xy \\ \dfrac{dy}{dt} = r_2 y\left(1 - \dfrac{y}{K_2}\right) - \alpha_2 xy, \quad \text{constants } r_1, r_2, K_1, K_2, \alpha_1, \alpha_2 > 0 \\ x(0) > 0,\ y(0) > 0. \end{cases}$$

(7 points) (a) Show that the solutions $(x(t), y(t))$ are positive for all $t > 0$.
(8 points) (b) Show that the solutions $(x(t), y(t))$ are bounded for all $t > 0$.
(5 points) (c) Show that the solutions $(x(t), y(t))$ exists for all $t > 0$.

2. Consider the IVP

$$\begin{cases} \dfrac{dx}{dt} = A(t)x \\ x(t_0) = x_0, \end{cases}$$

where $A(t)$ is a continuous $n \times n$ matrix function on $\mathbb{R}$.

(8 points) (a) Show that if

$$A(t)A(s) = A(s)A(t) \quad \text{for all } t, s \in \mathbb{R}, \tag{1}$$

then $x(t) = \left(\exp(\int_{t_0}^{t} A(s)ds)\right) x_0$, where $\exp(\int_{t_0}^{t} A(s)ds)$ is defined by it's power series.

(8 points) (b) Without the condition in (1), does the result in part (a) still hold? Give a proof if it holds or give a counterexample if it does not hold.

3. (7 points) (a) State the Liapunov theorem for stability and asymptotically stability for the equilibrium point $x = x_0$ of $x' = f(x)$, $x \in \mathbb{R}^n$.

(11 points) (b) Consider the system

$$\begin{cases} x_1' = -x_2^3 \\ x_2' = x_1^3. \end{cases}$$

Show that the the equilibrium point $(0, 0)$ is stable but not asymptotically stability. (**Hint.** Find a Liapunov function of the form $V(x_1, x_2) = ax_1^4 + bx_1^2x_2^2 + cx_2^4$.)

4. (20 points) Convert the system

$$\begin{cases} x' = y \\ y' = -x + \left(\dfrac{4 - x^2 - y^2}{4 + x^2 + y^2}\right) y. \end{cases}$$

into polar coordinates, draw the phase portrait, and find the ω-limit set for each trajectory. Make sure to justify the directions of the trajectories in your phase portrait. (**Hint.** $xx' + yy' = rr'$ and $(xy' - yx')/r^2 = \theta'$.)

(Please turn this page over and continue with Problem 5, Page 2.)

5. (2 points) (a) Give a definition of "topological conjugation" (or "topological conjugacy") for flows (solutions) φ^t and ψ^t in the Hartman-Grobman Theorem. Also, draw a diagram to give an explanation.

(7 points) (b) State the Hartman-Grobman Theorem.

(7 points) (c) (i) Consider the nonlinear system

$$\begin{cases} x_1' = x_1 + x_2^2, \\ x_2' = -x_2. \end{cases} \tag{2}$$

Derive the solution (the flow) $\varphi^t(x_1, x_2)$ of (2). Find unstable manifold $W^u(0)$ and stable manifold $W^s(0)$.

(2 points) (ii) Consider the linearized system of (2),

$$\begin{cases} x_1' = x_1, \\ x_2' = -x_2. \end{cases} \tag{3}$$

Derive the solution (the flow) $\psi^t(x_1, x_2)$ of (3). Find unstable subspace $E^u(0)$ and stable subspace $E^s(0)$.

(8 points) (iii) Find a homeomorphism h in the Hartman-Grobman Theorem for nonlinear system (2) and its linearized system (3). Justify your answer.