分析
(共2頁)

DEPARTMENT OF APPLIED MATHEMATICS
CHIAO TUNG UNIVERSITY
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Analysis
(TOTAL 100 PTS, two pages)

Throughout this exam, $\Phi \circ f(x) = \Phi(f(x))$, $\|x\|$ denotes the norm of x, $B(0;r) = \{x \in \mathbb{R}^n : \|x\| < r\}$, and $C[0,1]^*$ denotes the dual space of $C[0,1]$.

1. (50%) Prove or disprove the following statements:

   (a) Suppose μ is a finite Borel measure defined on $\mathbb{R}^n$. Then $f(r) = \mu(B(0;r))$ defines a right continuous function on $(0,\infty)$.

   (b) Let $g : (0,\infty) \mapsto \mathbb{R}$. If g is absolutely continuous on each finite subinterval of $(0,\infty)$, then $g' \in L^1(0,\infty)$.

   (c) Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^2[-\pi,\pi]$. Then $\lim_{n\to\infty} \int_{-\pi}^{\pi} f_n(x)dx$ exists.

   (d) Let $f \in C[0,1]$, $T \in C[0,1]^*$, and $\mathcal{P}$ denote the set of all complex polynomials. Then $\inf_{g\in\mathcal{P}} |T(f) - T(g)| = 0$.

   (e) Let f_n, g_n be Lebesgue measurable functions on $\mathbb{R}^3$ with $|f_n(x)| \le g_n(x)$ for all n and all x. If $g_n, g \in L^1(\mathbb{R}^3)$, $g_n(x) \to g(x)$ *a.e.*, $g_n \to g$ in $L^1(\mathbb{R}^3)$ and $f_n(x) \to f(x)$ *a.e.*. Then $\lim_{n\to\infty} \int_{\mathbb{R}^3} f_n(x)dx = \int_{\mathbb{R}^3} f(x)dx$.

2. (10%) Let $p > 1$, $t > 0$, $x > 0$. Suppose λ is $\sigma-$finite on $(0,\infty)$ and $\Lambda(x) := \lambda((0,x]) < \infty$. Prove that

$$(i) \quad \lambda(\Big\{y > 0 : \Lambda(y) \le t\Big\}) \le t; \qquad (ii) \quad \int_0^x \Lambda^{p-1} d\lambda \ge \frac{\Lambda^p(x)}{p}.$$

3. (10%) Let $1 < p < \infty$. Assume that $a_k \ge 0$ and $x_k \ge 0$ for all k. Prove that

$$\sum_{k=1}^{\infty} a_k x_k^{\frac{p-1}{p}} \le \Big(\sum_{k=1}^{\infty} a_k\Big)^{1/p} \Big(\sum_{k=1}^{\infty} a_k x_k\Big)^{\frac{p-1}{p}}.$$

4. (10%) Let $a_n \in \mathbb{C}$. Suppose $\sum_{n=1}^{\infty} a_n b_n$ converges for all $\{b_n\}_{n=1}^{\infty} \in \ell^2$. Prove that $\{a_n\}_{n=1}^{\infty} \in \ell^2$.

5. (10%) Let ν be a Borel measure on $\mathbb{R}$ with $\nu(\{0\}) = 0$. Define $\tilde{\nu}$ by $\tilde{\nu}(\Omega) = \nu(\Omega^{-1})$ for all Borel sets Ω, where $\Omega^{-1} = \{1/x : x \in \Omega\backslash\{0\}\}$. Prove that $\tilde{\nu}$ is also a Borel measure and

$$\int_{\Omega} f(x)d\nu(x) = \int_{\Omega^{-1}} f(1/y)d\tilde{\nu}(y)$$

for all Borel sets Ω and all nonnegative $\nu-$measurable functions f.

6. (10%) Let $k(x,t) \geq 0$, $k(x,t) \in C([0,1] \times [0,1])$, and

$$\int_0^1 k(x,t)dt = 1 \qquad \text{for all} \quad x \in [0,1].$$

Consider the operator $K : L^2[0,1] \mapsto L^2[0,1]$ defined by

$$\mathbb{K}f(x) = \int_0^1 k(x,t)f(t)dt \qquad (f \in C[0,1]).$$

Prove that $\|\Phi \circ \mathbb{K}f\|_2 \leq \|K\| \|\Phi \circ f\|_2$ for all $f \in C[0,1]$ and all nonnegative continuous convex function Φ on $\mathbb{R}$.