代數
(共2頁)

QUALIFYING EXAM

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Instructions. You need to show all your work in order to get full credit. When using a theorem, you must state it clearly and correctly.

Problem 1. *(10 points)* Let $\mathrm{Mat}_{n\times n}(K)$ denote the space of $n\times n$ matrices with coefficients in a field K. For $A\in \mathrm{Mat}_{n\times n}(K)$ define a linear map $T_A : \mathrm{Mat}_{n\times n}(K)\to \mathrm{Mat}_{n\times n}(K)$ by $T_A(B)=AB+BA$, for $B\in \mathrm{Mat}_{n\times n}(K)$. Express the trace of T_A as a function of the trace of A.

Problem 2. *(20 points)* Let R be a ring with 1. Suppose we have a sequence of left R-modules $\{M_i | i\in\mathbb{Z}\}$ and R-homomorphisms $\{d_i : M_i\to M_{i-1} | i\in\mathbb{Z}\}$ such that $d_i\circ d_{i+1}=0$, for all $i\in\mathbb{Z}$.

(i) Prove that $\mathrm{im} d_{i+1}\subseteq \ker d_i$, and hence $H_i=\ker d_i/\mathrm{im} d_{i+1}$ is an R-module.

Assume below that there exists a sequence of R-homomorphisms $\{s_i : M_{i-1}\to M_i | i\in\mathbb{Z}\}$ such that $d_i=d_i\circ s_i\circ d_i$, for all $i\in\mathbb{Z}$.

(ii) Prove that there exist R-submodules N_i and P_i of $\ker d_i$ such that $\ker d_i\cong N_i\oplus P_i$ with $N_i\cong H_i$, for every i. (Hint: Prove that $N_i=\ker(d_{i+1}\circ s_{i+1}+s_i\circ d_i)$.)

Problem 3. *(20 points)* Let G be a finite group.

(i) Suppose that $g^2=1$ for every $g\in G$. Prove that G is abelian.
(ii) Find all G with $|G|=8$ up to isomorphism.

Problem 4. *(25 points)* Let $f(x)=x^4-2x^2-2$.

(i) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
(ii) Prove that $\alpha=\sqrt{1+\sqrt{3}}$ and $\beta=\sqrt{1-\sqrt{3}}$ are roots of $f(x)$ in $\mathbb{C}$, and furthermore $\mathbb{Q}(\alpha)\cap\mathbb{Q}(\beta)=\mathbb{Q}(\sqrt{3})$. What are the other roots of $f(x)$?
(iii) Let F be a splitting field of $f(x)$ over $\mathbb{Q}$. What are the degrees of the field extensions $[F:\mathbb{Q}(\sqrt{3})]$ and $[F:\mathbb{Q}]$?
(iv) What is the Galois group of the extension $[F:\mathbb{Q}(\sqrt{3})]$?
(v) What is the Galois group of the extension $[F:\mathbb{Q}]$? Is it abelian?

Problem 5. *(25 points)* Let $\mathbb{F}_q$ denote the finite field of $q=p^m$ elements, where p is a prime. Let $\mathrm{GL}_n(\mathbb{F}_q)$ denote the group of invertible $n\times n$ matrices with coefficients in $\mathbb{F}_q$ and let $\mathrm{SL}_n(\mathbb{F}_q):=\{A\in \mathrm{GL}_n(\mathbb{F}_q) | \det A=1\}$.

(i) What is the order of the group $\mathrm{SL}_n(\mathbb{F}_q)$?

Below assume that $n=2$ and $p>2$. For an element $\xi\in\mathbb{F}_{q^2}$ we define $T_\xi:\mathbb{F}_{q^2}\to\mathbb{F}_{q^2}$ given by $T_\xi(z)=\xi z$, for $z\in\mathbb{F}_{q^2}$.

(ii) Identifying the $\mathbb{F}_q$-vector spaces $\mathbb{F}_{q^2}$ and $\mathbb{F}_q \oplus \mathbb{F}_q$ prove that the map $\xi \to T_\xi$ defines a group homomorphism $T : \mathbb{F}_{q^2} \setminus \{0\} \to \mathrm{GL}_2(\mathbb{F}_q)$.

(iii) Prove that there exists an element $\zeta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\zeta^q + \zeta = 0$. Conclude that $B_\zeta = \{1, \zeta\}$ is a $\mathbb{F}_q$-basis for $\mathbb{F}_{q^2}$.

(iv) Prove that, for $\xi \in \mathbb{F}_{q^2} \setminus \{0\}$, the matrix of the linear map T_ξ, with respect to the ordered basis B_ζ, is of the form $T_\xi = \begin{pmatrix} (\xi+\xi^q)/2 & \zeta(\xi-\xi^q)/2 \\ (\xi-\xi^q)/2\zeta & (\xi+\xi^q)/2 \end{pmatrix}$.

(v) Prove that $\mathbb{F}_{q^2} \setminus \{0\}$ has a unique subgroup $\mathbb{Z}_{q+1}$ of order $q+1$, and furthermore prove that $T_\xi \in \mathrm{SL}_2(\mathbb{F}_q)$, for all $\xi \in \mathbb{Z}_{q+1}$.