National Chiao Tung University
Department of Applied Mathematics
Discrete Mathematics Qualifying Examination
September 2012

Problem 1. Prove that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y = x$. (10%)

Problem 2. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a collection of m distinct k-subsets of $\{1, 2, \ldots, n\}$, where $k \leq n/2$, with the property that any two of the subsets have a nonempty intersection. Prove that $m \leq \binom{n-1}{k-1}$. (20%)

Problem 3. Using generating functions to solve $a_n + a_{n-1} - 2a_{n-2} = 2^{n-2}$, given that $a_0 = a_1 = 0$. (10%)

Problem 4. (a) Let G be some dependency graph for the events $A_1, \ldots, A_n$. Suppose that $\Pr(A_i) \leq p$, $i = 1, 2, \ldots, n$ and that every vertex of G has degree $\leq d$. Prove that if $4dp < 1$, then $\cap_{i=1}^{n} \overline{A_i} \neq \emptyset$. (20%)
(b) Prove that $N(k, k; 2) \geq c \cdot k \cdot 2^{k/2}$, where c is a constant. (10%)

Problem 5. If a graph G on n vertices has more than $\frac{1}{2}n\sqrt{n-1}$ edges, then G has girth ≤ 4. (10%)

Problem 6. Prove that if there exists an $S_\lambda(t, k, v)$ with $t \geq 2s$ and $v \geq k + s$, then we have $b \geq \binom{v}{s}$. (20%)