Qualifying Exam.
Ordinary Differential Equations September 2012

This exam. contains 5 problems with a total of 100 points. Do all 5 problems. Show all your work to get full credits.

1. (**Gronwall's inequality**) Let $\lambda(t)$ be a real continuous function and $\mu(t)$ a nonnegative continuous function on $[a,b]$. Assume that a continuous function $y(t)$ satisfies

$$y(t) \le \lambda(t) + \int_a^t \mu(s)y(s)ds, \quad a \le t \le b.$$

(14 points) (a) Show that

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s)\exp\left(\int_s^t \mu(\tau)d\tau\right) ds, \quad a \le t \le b.$$

In particular if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \le \lambda \exp\left(\int_a^t \mu(s)ds\right), \quad a \le t \le b.$$

(6 points) (b) **(Local uniqueness)** Let f and $\partial f/\partial y$ be continuous in some rectangle R: $|t-t_0| \le c$, $|y-y_0| \le d$. Apply Gronwall's Inequality to prove that there is some interval $|t-t_0| \le h \le c$ in which there exists **at most one** solution of the IVP

$$y' = f(t,y), \quad y(t_0) = y_0.$$

2. (**Variation of constant formula**) Let $\Phi(t)$ be a fundamental matrix of

$$\frac{dx}{dt} = A(t)x$$

and $\varphi(t,\tau,\xi)$ be the solution of the IVP

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + g(t), \\ x(\tau) = \xi, \end{cases} \tag{1}$$

where $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ and $g(t) \in \mathbb{R}^n$ are continuous.

(5 points) (a) Find a constant matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$\varphi(t,\tau,\xi) \equiv \Phi(t)\Lambda\xi + \int_\tau^t \Phi(t)\Phi^{-1}(\eta)g(\eta)d\eta \tag{2}$$

is the solution of (1).

(5 points) (b) With the constant matrix Λ you find in (a), prove that $\varphi(t,\tau,\xi)$ in (2) is the solution of (1).

(Please continue to do Problem 2, parts (c)-(d) and Problems 3-5.)

(5 points) (c) When $A(t) \equiv A$ is a constant matrix, solve the IVP

$$\begin{cases} \dfrac{dx}{dt} = Ax + g(t), \\ x(\tau) = \xi. \end{cases}$$

(5 points) (d) Solve the IVP

$$\begin{cases} \dfrac{dx}{dt} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}, \\ x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{cases}$$

3. (20 points) Consider the nonlinear spring motion with friction

$$x''(t) + f(x)x'(t) + g(x) = 0,$$

where $f, g \in C$, $f(x) > 0$, $x \neq 0$, $xg(x) > 0$, $x \neq 0$, $g(0) = 0$, and $G(x) = \int_0^x g(s)ds \to \infty$ as $|x| \to \infty$. Prove that $(x(t), x'(t)) \to (0,0)$ as $t \to \infty$. (**State the theorem used precisely**.)

4. Consider the following Predator-Prey system

$$\begin{cases} \dfrac{dx}{dt} = x\left(\gamma\left(1 - \dfrac{x}{K}\right) - \alpha y\right), \\ \dfrac{dy}{dt} = (\beta x - d - \delta y)\, y, \\ x(0) > 0,\ y(0) > 0 \end{cases}$$

where constants $\gamma, K, \alpha, \beta, d, \delta > 0$ and $\beta K > d$.

Do the followings (i)-(iv):

(4 points) (i) Find all equilibria with nonnegative components.
(10 points) (ii) Do stability analysis for each equilibrium.
(2 points) (iii) Find the stable and unstable manifolds of each saddle point.
(4 points) (iv) Predict the global asymptotic behavior and sketch the phase portrait.

5. (20 points) Prove that the system

$$\begin{cases} x' = x + y - x(x^2 + 2y^2), \\ y' = -x + y - y(x^2 + 2y^2) \end{cases} \tag{3}$$

has a limit cycle inside an annular region $D = \{(x,y) : a < x^2 + y^2 < b\}$ for some $a, b > 0$ satisfying

$$0 < b - a < 1.$$

Find a feasible pair (a, b). Also sketch the phase portrait of (3) on the (x,y)-plane. (**State the theorem used precisely**.)