國立交通大學應用數學研究所博士班資格考試

科目：代數

2012 年 2 月 24 日

1. (% 20) Let G be a finite group. Recall that a simple group is a group without proper, non-trivial normal subgroup.
   (a) Assume that G has a subgroup H of index $\ell > 1$ such that $|G| \nmid \ell!$ ($|G|$ denotes the order of G). Show that G is not a simple group.
   (b) Suppose that the order of G is $5808 = 2^4 \times 3 \times 11^2$. Can G be a simple group? Explain your answer.

2. (% 10) Let G be a subgroup of the symmetric group S_n. Assume that G contains an odd permutation. Prove that the index $[G : G \cap A_n] = 2$.

3. (% 10) Let D be a principal ideal domain (PID). Show that a non-zero ideal $\mathcal{P}$ of D is a prime ideal if and only if $\mathcal{P}$ is a maximal ideal.

4. (% 15) Let k be a field and let $A \in M_n(k)$ be an $n \times n$ matrix. Let $R = k[A]$. Note that you need to explain your answer otherwise there won't be any credit for your answer.
   (a) Show that R is a finite dimensional vector space over k. Is it always true that $\dim_k R = n$? If yes, give a proof; otherwise, give a necessary and sufficient condition so that $\dim_k R = n$.
   (b) In general, R may not be a field. Give a necessary and sufficient condition such that R is a field.

5. (% 20) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be a linear transformation give by the following formula

$$T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{pmatrix} 54 & 74 & 62 & 66 \\ 294 & 404 & 338 & 360 \\ 48 & 68 & 56 & 60 \end{pmatrix}, \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

   Note that A is a matrix with integer entries, therefore, $M := T(\mathbb{Z}^4)$ is a $\mathbb{Z}$-submodule of $\mathbb{Z}^3$.
   (a) Find a basis of the $\mathbb{Z}$-module M.
   (b) Let $N = \ker T \cap \mathbb{Z}^4$. Then, N is a $\mathbb{Z}$-submodule of $\mathbb{Z}^4$. Find a basis of N.
   (c) Determine the structure of $\mathbb{Z}^3/M$.

6. (% 10) Let K be a field of positive characteristic p containing the finite field $\mathbb{F}_q$ of q-elements with $q = p^n$ for some n. We say that a polynomial $f(x) \in K[x]$ is $\mathbb{F}_q$-linear if it satisfies the following two conditions

   (i) $f(x+y) = f(x) + f(y)$ for all $x, y \in \bar{K}$ (algebraic closure of K),

   (ii) $f(\zeta x) = \zeta f(x)$ for all $x \in \bar{K}$ and $\zeta \in \mathbb{F}_q$.

   Let $P(x) \in K[x]$ be a separable polynomial and let $W = \{w_1, \ldots, w_r\}$ $(r = \deg(P))$ be the set of roots of P in $\bar{K}$. Prove that P is $\mathbb{F}_q$-linear if and only if W is a finite dimensional vector space over $\mathbb{F}_q$.

7. (% 15) Let $K = \mathbb{C}(t)$ be the field of rational function in one variable and let $f(X) = X^3 - 2tX + t \in K[X]$.

   (a) Show that $f(X)$ is irreducible over K and that the Galois group of $f(X)$ over K (meaning the Galois group of the splitting field of $f(X)$ over K) is isomorphic to S_3.

   (b) Is it true that for $t = a$, a negative square free integer, the polynomial $f_a(X) = X^3 - 2aX + a$ is irreducbile over $\mathbb{Q}$ and the Galois group of f_a over $\mathbb{Q}$ is also isomorphic to S_3?