DEPARTMENT OF MATHEMATICS
CHIAO TUNG UNIVERSITY
Ph. D. Qualifying Examination
September, 2011
Analysis (分析)
(TOTAL 100 PTS)

(共2頁)

Throughout this exam, $|\cdot|$ and dx denote the Lebesgue measure.

1. (50%) Prove or disprove the following statements:

   (a) Let $A \subset \mathbb{R}^2$ with $|A| = \infty$. Then
   $$A \cap \{(x, y) \in \mathbb{R}^2 : n < x^2 + y^2 < n + 1\} \neq \emptyset$$
   holds for infinitely many integers n.

   (b) Assume that $f : [0, 1] \mapsto \mathbb{R}$ is of bounded variation. Then the equation $|\{x \in [0, 1] : f \text{ is continuous at } x\}| = 1$ always holds.

   (c) There does not exist $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ such that
   $$\lim_{T\to\infty} \int_{-T}^{T} f(x)dx = 0 \quad \text{and} \quad \lim_{T\to\infty} \int_{-T}^{2T} f(x)dx = -1.$$

   (d) There exists a sequence $\{f_n\}$ in $L^2[-\pi, \pi]$ such that $f_n \longrightarrow f$ in $L^2[-\pi, \pi]$ for some f, but f_n diverges in $L^1[-\pi, \pi]$.

   (e) Let $T(f) = \int_0^1 \sqrt{x} f(x)dx$. Then T defines a bounded linear functional on $L^p[0, 1]$ and $\|T\| = (2/(q+2))^{1/q}$, where $1 < p < \infty$ and $1/p + 1/q = 1$.

2. (10%) Let $f_n \in C[a, b]$ for $n = 1, 2, \cdots$, where $0 < a < b < \infty$. Suppose that the derivatives f_n' exist and are uniformly bounded on $[a, b]$.

   (a) Prove that $\{f_n\}$ is equicontinuous on $[a, b]$.

   (b) Does f_n have a uniformly convergent subsequence? Verify your answer.

3. (10%) Let μ be a positive Borel measure on $\mathbb{R}$ such that
   $$|\mu(A)| \leq \int_A \frac{dx}{1 + x^2} \qquad \text{for all Borel subsets } A \text{ of } \mathbb{R}.$$
   Prove that there exists some $f \in L^1(\mathbb{R})$ such that $0 \leq f(x) \leq \frac{1}{1+x^2}$ for almost all $x \in \mathbb{R}$ and
   $$\mu(A) = \int_A f(x)dx \qquad \text{for all Borel subsets } A \text{ of } \mathbb{R}.$$

4. (10%) Let $\lambda > 0$ and F be a closed subset of $(0, 1)$. Prove that

$$\int_F \left(\int_0^1 \frac{\delta(y)^\lambda}{|x-y|^{1+\lambda}} dy \right) dx \le \frac{2}{\lambda} |(0,1) \setminus F|,$$

where $\delta(y)$ denotes the distance from y to F.

5. (10%) Assume that f and g are two absolutely continuous functions on $[a, b]$.

   (a) Prove that fg is also absolutely continuous on $[a, b]$.

   (b) Can you conclude that

$$\int_a^b f(t)g'(t)dt + \int_a^b f'(t)g(t)dt = f(b)g(b) - f(a)g(a)?$$

   Verify your answer.

6. (10%) Let $\alpha > 0$ and μ be a positive Borel measure defined on $[0, 1]$. Suppose that $\mu([0, 1]) = 1$ and

$$\int_0^1 \theta^k d\mu(\theta) = \frac{\alpha}{k+\alpha} \qquad (k = 0, 1, 2, \cdots).$$

   Prove that

$$\int_0^1 f(\theta)d\mu(\theta) = \alpha \int_0^1 \theta^{\alpha-1} f(\theta)d\theta \quad \text{for all } f \in C[0, 1].$$