Ph.D. entrance exam, 2011 – Linear Algebra

Notations.

1. Throughout the exam, the letter n will always represent a positive integer.
2. The letter $\mathbb{R}$ denotes the field of real numbers. Hence, the notation $\mathbb{R}^n$ represents the usual Euclidean space of dimension n. The notation $M_n(\mathbb{R})$ stands for the set of $n \times n$ matrices over $\mathbb{R}$.
3. The identity matrix of size n is denoted by I_n.
4. For a matrix A, we let A^t denote the transpose of A. For a square matrix B, we let $\operatorname{tr} B$ and $\det B$ denote the trace and the determinant of B, respectively. If a square matrix C is nonsingular, then C^{-1} denotes its inverse.
5. For a square matrix A, we let $\exp A = \sum_{n=0}^{\infty} A^n/n!$.

Problems.

1. Define an equivalence relation $\sim$ on $M_n(\mathbb{R})$ by $A \sim B$ if and only if there exists an invertible matrix P in $M_n(\mathbb{R})$ such that $A = P^{-1}BP$.
   (a) Recall that a matrix A in $M_n(\mathbb{R})$ is said to be *unipotent* if $(A - I_n)^k = 0$ for some positive integer k. Determine the number of equivalence classes of unipotent matrices in $M_6(\mathbb{R})$. **(10 points.)**
   (b) Recall that a matrix B in $M_n(\mathbb{R})$ is said to be *nilpotent* if $B^k = 0$ for some positive integer k. Prove that if $B \in M_n(\mathbb{R})$ is nilpotent, then $\exp B$ is unipotent. **(10 points.)**
2. Let $A \in M_n(\mathbb{R})$.
   (a) Prove that if A is skew-symmetric, then $\exp A$ is orthogonal. **(10 points.)**
   (b) Prove that if $\operatorname{tr} A = 0$, then $\det(\exp A) = 1$. **(15 points.)**
3. Let $V = M_n(\mathbb{R})$. Define $f : V \times V \to \mathbb{R}$ by
$$f(A, B) = \operatorname{tr}(AB).$$
   (a) Prove that f is a symmetric bilinear form on V. (*Symmetric* means $f(A, B) = f(B, A)$ for all $A, B \in V$. *Bilinear* means f is linear in each of the two variables.) **(10 points.)**
   (b) Let U be the subspace of V consisting of symmetric matrices and W be the subspace of V consisting of skew-symmetric matrices. Prove that U and W are orthogonal components of each other with respect to the symmetric bilinear form f. That is,
$$\{A \in V : f(A, B) = 0\ \forall B \in U\} = W,$$
$$\{A \in V : f(A, B) = 0\ \forall B \in W\} = U.$$
   (*Hint*: Choose suitable bases for U and W first.) **(15 points.)**
4. Let V be the vector space $\mathbb{R}^n$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$. Recall that a linear transformation $T : V \to V$ is said to be an *isometry* if $\langle Tv, Tv \rangle = \langle v, v \rangle$ for all $v \in V$.
   (a) Prove that every isometry has determinant ± 1. **(10 points.)**
   (b) Prove that if n is odd, then every isometry T with determinant 1 fixes some nonzero vector in $\mathbb{R}^n$. (That is, $Tv = v$ for some nonzero vector v.) **(15 points.)**
   (c) Give an example of an isometry of determinant 1 for $\mathbb{R}^2$ that does not fix any nonzero vector. **(5 points.)**